AdS/CFT Correspondence and Geometry

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by
Palla Shubhang Sharma
PHYS 11202013034

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To all the relationships we hold…especially to my parents.

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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The thesis work reported in the thesis entitled ....
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School:
Date:

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ABSTRACT

The AdS/CFT correspondence proposes a duality between a gravitational theory in a (d+1)-dimensional anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) living on its d-dimensional boundary. This work introduces the foundational ingredients of the correspondence through a detailed study of the geometry of AdS spacetimes and the symmetry structures of conformal field theories, particularly in two dimensions, where enhanced symmetry allows sometimes exact results. On the gravity side, the focus lies on maximally symmetric solutions to Einstein's equations with negative cosmological constant, coordinate charts, and boundary structure. On the field theory side, conformal symmetry, primary fields, correlation functions, Ward identities, and the operator formalism are developed in detail.

A key theme of the thesis is the geometric underpinning of the correspondence, with an emphasis on the role of Riemannian and conformal geometry. In particular, the Fefferman–Graham ambient metric construction is presented as a formal geometric justification for this holographic setup. The AdS/CFT dictionary is explored via the GKP–Witten prescription and the behaviour of bulk fields near the boundary. Applications include the study of strongly coupled gauge theories and black hole thermodynamics using dual weakly coupled gravitational theories.

This project, while motivated by physics, is approached with a mathematical inclination, seeking to understand the geometrical content underlying AdS/CFT and to place the correspondence within the broader interactions of geometric structures in mathematical physics.

Contents

Chapter 1

Introduction

Holography is the idea that physics in $d + 1$ dimensions can be understood by a theory at infinity (a '*hologram* at infinity') in d or fewer dimensions. In the context of AdS/CFT, it is conjectured that the AdS bulk gravity theory is related to a conformal field theory (CFT) on the boundary. Spaces like spheres and planes are heavily studied because they possess *maximal symmetry* where the number of isometries (metric-preserving transformations) is maximum. One such generalisation of these spaces is the negatively curved *Ads* space, which also solves Einstein's equations in a vacuum. It has been very fruitful to consider this space in string theoretical studies. On the other hand, we have scale invariance, which is a subgroup of the larger symmetry of *conformal invariance*. Conformal transformations are essentially angle-preserving transformations, i.e. they affect the metric only up to a scale factor. This symmetry, for example, is seen in many statistical systems near the critical point, and is the main content of conformal field theories. The goal of this thesis has been to understand fundamental elements behind AdS/CFT Correspondence, namely, the geometry of Ads spacetime, Conformal field theory (CFT), the prescription to relate quantum theories in the bulk and boundary of AdS, along with some geometric aspects of manifolds in the context of AdS/CFT.

Chapter 2 introduces formally the Riemannian structure on smooth manifolds and fundamental geometric tools like connections and curvature. Towards the end, we identify one of the crucial conformal contents of the Riemannian geometry - the Weyl Tensor (and the Cotton tensor). In the context of AdS/CFT, as we will see, conformal geometry is in particular relevant, where the objects are manifolds that possess a class of Riemannian metrics rather than a unique one. We discussed this connection in the last chapter. Chapter 3 exposes the Anti-de Sitter space in all its symmetries and coordinate forms of the metric. Importantly, there is the notion of (conformal) boundary structure of Ads. With this, studying Conformal Field theory becomes natural on the boundary, and chapters 4,5,6 discuss the structure of this theory in general and 2 dimensions. CFTs in 2 dimensions are very powerful with higher symmetries and solvable (sometimes exactly) in nature. Given an understanding of conformal transformations in 2 dimensions, we relate these back to isometries on AdS_3 in chapter 7. Chapter 8 lays out the conjecture via two proposed prescriptions along with the subtleties that occur in Lorentzian signatures to compute the correlation functions on the boundary CFT.

Central to the geometric interaction with AdS/CFT is the theory of conformal invariants which is the subject of Chapter 9. The ambient metric construction due to Fefferman and Graham (FG) is also discussed, which laid out the foundations in conformal geometry and has also played an important role in the development of AdS/CFT holography. It's the formal underlying reason why AdS/CFT correspondence makes sense.

Guiding Principles

> *In this drama of mathematics and physics, which fertilise each other in the dark, but which prefer to deny and misconstrue each other face to face—I cannot, however, resist playing the role of a messenger, albeit, as I have abundantly learned, often an unwelcome one.* (Hermann Weyl (1928))

Now, nearly a century later, with the advent of gauge theory, string theory, twistor theory, etc., the interaction between mathematics and physics is more ripe than ever. After a seemingly large evolution of both the fields independently and divergently, the late 20th century has brought back the interactions beyond the fundamentals as we now know them. Wide areas of mathematics and Physics have now remarkable connections in geometry (algebraic and differential), topology and even number theory. It is in this spirit that some of the work in this thesis is carried out. After a semester and a half work on understanding AdS, CFT and AdS/CFT correspondence from the physics point of view, the goal was shifted to connect these areas to mathematical literature considering the author's current interests to pursue research in mathematics. Any such effort to connect AdS/CFT to mathematics involves an understanding of the basics of the Geometry of Manifolds. Thus, most of the work after the shift in goals had gone into understanding the geometric tools. The existing connection in this setting is discussed briefly in the last chapter.

Chapter 2

Riemannian Geometry

The geometry in 2-dimensions is fully established, chiefly due to Gauss. Riemann laid out the foundations for studying curved geometry in higher dimensions, which has played a revolutionary part in understanding gravitational physics, along with establishing modern differential geometry. The inner product on the Euclidean space $\mathbb{R}^n$ (the dot product) is the primary tool to probe the geometry. The starting point is the generalisation of this notion to manifolds by smoothly assigning an inner product to the tangent space at each point.

Definition 2.0.1. *A Riemannian manifold (M, g) is a smooth manifold M with a Euclidean inner product g_p on each tangent space T_pM of M. We assume that the map $p \mapsto g_p$ is smooth i.e for all vector fields $X, Y \in \mathfrak{X}(M)$, $g_p\left(X_p, Y_p\right)$ is smooth for all $p \in M$.*

So, g is essentially a smooth 2-tensor field of the tensor bundle $T^{(0,2)}(TM)$ (usually written $g \in \mathcal{T}^2(M)$) whose value g_p for all $p \in M$ is a symmetric, bi-linear and positive definite 2-tensor on T_pM (i.e. an inner product of T_pM). It is referred to as *Riemannian metric* or simply metric.

All manifolds are locally Euclidean. All Riemannian manifolds are also pointwise Euclidean in the sense that any inner product spaces of the same dimension (here T_pM) are isometric to each other (due to Gram-Schimdt orthogonalization) and thus to $\mathbb{R}^n$ with the canonical inner product.

Example 2.0.1. $(\mathbb{R}^n, g_{\mathbb{R}^n})$ is a Riemannian Manifold. The tangent bundle $T\mathbb{R}^n$ is just $\mathbb{R}^n \times \mathbb{R}^n$ and we can use the standard inner product to define the Riemannian structure g,

$$g_{\mathbb{R}^n}\left((p, v), (p, w)\right) = v \cdot w.$$

Definition 2.0.2. *A Riemannian isomtery between Riemannian manifold (M, g_M) and (N, g_N) is a diffeomorphism $F : M \to N$ such that $F^* g_N = g_M$, i.e.*

$$F^* g_N(v, w) := g_N (F_*(v), F_*(w)) = g_M (v, w)$$

for all tangent vectors $v, w \in T_P M$ and $p \in M$. In this case, F^{-1} is also an isometry.

Example 2.0.2. Any finite-dimensional vector space V (which has a manifold structure by decalring the canonical map $\phi : V \to \mathbb{R}^n$ to be a homeomorphism) can be made into a Riemannian manifold with the following canonical metric,

$$g\left((p, v), (p, w)\right) = v \cdot w$$

where we have used $T_w V \cong V$ for all $w \in V$.

Any two such Riemannian manifolds $(V, g_V), (W, g_W)$ of same dimension are isometric! This is because there is always a linear isometry $F : V \to W$ between V and W which is easily seen to be a Riemannian isometry. So, $(\mathbb{R}^n, g_{\mathbb{R}^n})$ is the only Riemannian manifold of the above type (upto isometries).

Let M is a manifold and (N, g_N) is a Riemannian manifold. If $F : M \to N$ is an immersion (or an embedding) then it is possible to take pullback of the metric that defines an inner product on the tangent spaces of M,

$$g_M (v, w) = g_N (F_*(v), F_*(w)) .$$

This is an inner product because $g_M (v, v) = 0 \iff F_*(v) = 0 \iff v = 0$. Note that the above push forward vector fields is just an abuse of notation. Pushforwards of vector fields are defined under special conditions (diffeomorphism or F-relatedness). But since pullback of forms is always defined, it is tempting to write pointwise pushforward of tangent vectors as the pushforward of the vectorfield itself. We don't need a smooth vector field on M to define the form on M.

We thus have,

Definition 2.0.3. *A Riemannian immersion (embedding) is an immersion (embedding) $F : M \to N$ such that $g_M = F^* g_N$. They are also called isometric immersions.*

Example 2.0.3. Let's consider $S^n(R) = \{x \in \mathbb{R}^{n+1} \mid |x| = R\}$, the Euclidean sphere of radius R. We can induce a metric on S^n by embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. This is the canonical metric on S^n.

Let's debunk the *isometry* in isometric immersions.

Between $\mathbb{R}^k$ and $\mathbb{R}^n$ there are immersions like

$f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$ or $f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 1, \ldots, -1)$ and several more which are all also isometries!

But there are also other immersions.

1. Unit speed curve $c : \mathbb{R} \to \mathbb{R}^2, |\dot{c}(t)| = 1$ is a Riemannian immersion,

$$\begin{aligned} c^* g_{\mathbb{R}^2}\left(\alpha, \beta\right) &= g_{\mathbb{R}^2}\left(\alpha \dot{c}(t), \beta \dot{c}(t)\right) \\ &= \alpha\beta g_{\mathbb{R}^2}\left(\dot{c}(t), \dot{c}(t)\right) \\ &= \alpha\beta = g_{\mathbb{R}}\left(\alpha, \beta\right). \end{aligned}$$

   which need not be distance preserving! For example consider,

$$t \mapsto (\cos t, \sin t)$$

   the (Riemannian) immersion of $\mathbb{R}$ onto S^1. Similarly, consider an embedding, $\mathbb{R} \hookrightarrow \mathbb{R}^2$ as follows:

$$t \mapsto \left(\log\left(t + \sqrt{1+t^2}\right), \sqrt{1+t^2}\right)$$

2. The above examples allows us to construct a more general class of isomteric immersions which are not distance preserving. For example,

$$F : \mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$$
$$(x^1, \ldots, x^k) \longmapsto F\left((x_1, \ldots, x^k)\right) = (c(x^1), x^2, \ldots, x^k).$$

where c is the unit speed curve from $\mathbb{R}$ to $\mathbb{R}^2$. We can realize cylinder as a Riemannian immersion in $\mathbb{R}^3$ something like a half cylinder (a plane bent in a direction) as an embedding in $\mathbb{R}^3$. These are not distance preserving maps but in the sense of curvature these are just same as $\mathbb{R}^2$. *A cylinder can be unrolled into a plane.* They have a zero Gaussian curvature! Much of the efforts soon would be to make such arguments in higher dimensions and on abstract spaces.

There is also a concept of Riemannian Submersions, which we will skip.

Definition 2.0.4. *A semi- or pseudo-Riemannian manifold is a maniofl with smoothly varying symmetric, bi-linear and non-degerate 2-form g on each tangent space.*

By non-degenrate, we mean: For each $v \in T_pM$ there exists $w \in T_pM$ such that $g(v, w) \neq 0$. This is generalization of the Riemannian metric where we had $g(v, v) > 0$ for $v \neq 0$ (positive definite) satisfying the above condition.

Lemma 2.0.1. *Each tangent space T_pM of a semi-Riemannian manifold (M, g) admits a decomposition,*

$$T_pM = P \oplus N$$

such that g is positive definite on P and negative definite on N. These subspaces are not unique but their dimension is well-defined.

Definition 2.0.5. *The index of a connected semi-Riemannian manifold is defined as the dimension of the subspace N on which g is negative definite.*

Example 2.0.4. Let $n = n_1 + n_2$ and $\mathbb{R}^{n_1,n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. There is a trivial decomposition, $\mathbb{R}^{n_1,n_2} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$. Over which we can define a semi-Riemannian metric of index n_2 as,

$$g\left((p,v),(p,w)\right) = v_1 \cdot w_1 - v_2 \cdot w_2$$

Minkowski space-time is a 1-index semi-Riemannian manifold $\mathbb{R}^{n,1}$.

Example 2.0.5. The family of Hyperbolic spaces $H^n(R) \subset \mathbb{R}^{n,1}$.

The hyperboloids embedded in $\mathbb{R}^{n,1}$ are defined by,

$$(x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = -R^2.$$

We denote the branch with $x^{n+1} > 0$ by $H^n(R)$.

Infact $\mathbb{R}^{n+1}$ induces a Riemannian metric on $H^n(R)$. If $v = (v^1, \ldots, v^n, v^{n+1}) \in T_pH^n(R)$, $p \in H^n(T)$ then since $H^n(R)$ is a level set, we have

$$v^1 p^1 + \cdots + v^n p^n - v^{n+1} p^{n+1} = 0.$$

So,

$$g_{H^n(R)}(v,v) = \left(v^1\right)^2 + \cdots + (v^n)^2 - \left(v^{n+1} p^{n+1}\right)^2.$$

and one can show using Cauchy-Shwarz that this is positive definite. $H^n \equiv H^(1)$ is usually called the hyperbolic n-space.

Note that in Euclidean space, we could write the volume of parallelopiped formed by n-vectors as,

$$vol(v_1, \ldots, v_n) = det[g(v_i, e_j)] = det[v_1, \ldots, v_n]$$

which is valid over any other positively oriented orthonormal basis.

Definition 2.0.6. *We define the volume form for any oriented Riemannian manifold* (M, g) *to be the n-form given by,*

$$vol_g(v_1, \ldots, v_n) = vol(v_1, \ldots, v_n) = det[g(v_i, e_j)]$$

where $e_1, \ldots, e_n$ *is any positively oriented orthonormal basis. If the manifold is not oriented or orientable then we can use* $E_1, \ldots, E_n$ *an orthogonal local frame on* M *and declare it to be positive and define the volume form locally by,*

$$vol(X_1, \ldots, X_n) = det[g(X_i, E_j)].$$

The height of arbitrary vector X *in the ith coordinate is given by* $vol(E_1, \ldots, X, \ldots, E_n) = g(X, E_i)$.

2.1 Isometry Groups

For a Riemannian manifold (M, g) we denote the group of Riemannian isometries $F : (M, g) \to (M, g)$ by $Iso(M, g)$ or $Iso(M)$

Definition 2.1.1. *The isotropy at* p *denoted by* $Iso_p(M, g)$ *is the (stabilizer) subgroup of* $Iso(M, g)$ *such that* $F(p) = p$*.*

It is easy to see that the isotropy is a subgroup of Iso. The differential of any isometry will be an orthogonal matrix (locally) which form a subgroup of $GL(n, \mathbb{R}^n)$.

Definition 2.1.2. *A Riemannian manifold is said to be homogenous if its isometry group acts transitively, i.e. for any* $p, q \in M, \exists F \in Iso(M, g)$ *such that* $F(p) = q$*.*

Example 2.1.1. The isometry group of the Euclidean space $\mathbb{R}^n$ is given by,

$$Iso(M,g) = \mathbb{R}^n \rtimes O(n) = \{F : \mathbb{R}^n \to \mathbb{R}^n \mid F(x) = v + Ox, v \in \mathbb{R}^n, O \in O(n)\}$$

$\Longleftarrow$: Note that F is a difeeomorphism. And the differential map F_* is just the orthogonal matrix $O \in O(n)$, which preserves the inner product.

$\Longrightarrow$: Suppose F is an arbitrary isometry. Then $G(x) = F(x) - F(0)$ also defines an isometry (the differential is unchanged). Consider also the map $G_{*0}(x) = G_{*0}x, x \in \mathbb{R}^n$. $G_{*0} \in O(n)$, since $F_{*0} \in O(n)$. So, G and G_* are both Riemannian Isometries with $G(0) = G_{*0}(0)$ and same differentials at 0. By a uniqueness result of Riemannian isometries, $G = G_{*0}$. Thus $Iso_0(\mathbb{R}^n) = O(n)$. Same is true for any point $p \in \mathbb{R}^n$. And $F(x)$ is thus of the form $F(0) + Ox$, for some $F(0) \in \mathbb{R}^n, O \in O(n)$. So, $\mathbb{R}^n \simeq Iso/Iso_p$!

Lemma 2.1.1. *Any homogenous space* $M = Iso(M)/Iso_p(M)$.

Proof. At the level of sets, this is clear. Just map $[F] = \{G \in Iso \mid F - G \in Iso_p\} \in Iso/Iso_p \mapsto F(p) \in M$ which is well defined. And since for any $q \in M$ (and given $p \in M$), we have by homogenity, $F \in Iso$ such that $F(p) = q$. So map $q \mapsto [F]$. ◻

Example 2.1.2. For spheres $S^n(R) \subset \mathbb{R}^{n+1}$,

$$Iso\left(S^n(R), g_{S^n(R)}\right) = O(n+1) = Iso_0\left(\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}}\right)$$

$\Longleftarrow$: Since we induce the metric on $S^n(R)$ from $\mathbb{R}^{n+1}$, $O(n+1)$ being the isometry group of $\mathbb{R}^{n+1}$ is also an isometry subgroup of $S^n(R)$.

$\Longrightarrow$: Consider an arbitrary isometry $F \in Iso(S^n(R))$. We can construct an orthogonal matrix out of this isometry,

$$O = \left[\frac{1}{R}F(Re_1)\ F_{*Re_1}(e_2)\ \cdots\ F_{*Re_1}(e_{n+1})\right].$$

To see this, note:

- $T_{Re_1}S^n = F(Re_1)^\perp$. So, $F_{*e_1}(e_i) \perp F(Re_1)$.
- $\{e_i\}_{i=2,\dots,n+1}$ forms a basis for $T_{Re_1}S^n(R)$ for the same reason as above. And since F_* is a linear isometry, $\{F_{*e_1}(e_i)\}_{i=2,\dots,n+1}$ forms an orthonormal basis for $T_{F(Re_1)}S^n(R)$.

So, $O \in O(n+1)$ and defines a Riemannian Isometry satisfying $O(Re_1) = F(Re_1)$ and $F_{*Re_1} = O_{Re_1} = O$. Thus by uniqueness result, we must have $F = O \in O(n+1)$.

Here, we find that $Iso_p(S^n(R)) \cong O(n)$ for any $p \in S^n(R)$. And $S^n \simeq O(n+1)/O(n)$.

Example 2.1.3. Similarly for the Hyperbolic spaces $H^n(R)$ one can show that,

$$Iso(H^n(R)) = O^+(n,1)$$

where $O^+(n,1)$ is the subgroup of $O(n,1) = \left\{L : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1} \mid g(Lv,Lv) = g(v,v)\right\}$ which preserve the condition $x^{n+1} > 0$. The isotropy group is again given by $O(n)$, and $O^+(n+1)$ acts on $H^n(R)$ transitively. So, $H^n(R) = O^+(n+1)/O(n)$.

2.2 Local Expression of Metrics

For a Riemannian manifold (M,g), if $X-1,\dots,X_n$ forms a local frame over an open set U of M, we can use the dual frame (coframe) $\sigma^1,\dots,\sigma^n$ to represent the metric locally as,

$$\begin{aligned} g(v,w) &= g\left(\sigma^i(v)X_i, \sigma^j(v)X_j\right) \\ &= g\left(X_i, X_j\right)\sigma^i\sigma_j. \end{aligned}$$

We have identified the components of $v = v^iX_i$ as $v^i = \sigma^i(v)$ usign the dual basis.

We thus have the following frame representation of a metric,

$$g = g\left(X_i, X_j\right) \sigma^i \otimes \sigma x^j = g_{ij}\, sigma^i \sigma^j.$$

This defines a symmetric positive definite matrix $[g_i j]$.

In particular, on any chart $(U, x_1, \ldots, x_n)$ we can use the coordinate 1-forms $dx_1, \ldots, dx_n$ which forms a local frame of M to write the metric as

$$g = g\left(\partial_i, \partial_j\right) dx^i \otimes dx^j = g_{ij} dx^i dx^j$$

Example 2.2.1. The canonical metric on $\mathbb{R}^n$ in the identity chart can be expressed as,

$$g = \delta_{ij} dx^i dx^j = \sum_{i=1}^{n} \left(dx^i\right)^2 .$$

Example 2.2.2. On $\mathbb{R}^2$ we can write the metric in polar charts in a region excluding a half-line (say positive X-axis),

$$g = dr^2 + r^2 d\theta^2.$$

This follows immediately from the relation with cartesian charts, $x = r\cos\theta$, $y = r\sin\theta$. and $g = dx^2 + dy^2$.

Example 2.2.3 (Surface of revolution in $\mathbb{R}^3$)**.** Starting with an injective profile curve $c : I \overset{open}{\subset} \mathbb{R} \to \mathbb{R}^3$ given by

$$c(t) = (r(t), 0, z(t)),$$

where $r(t) > 0$, we can define the surface of revolution via,

$$(t, \theta) \mapsto f(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

Denote, $X = f(\mathbb{R}^2)$

In neighborhoods of points where f is an immersion, we can pushforward the vector fields (since f is injective), $\partial_t, \partial_\theta$ to X defining a natural dual local frame, $dt, d\theta$ on X. We can then induce the metric on X in this frame using the Euclidean metric $dx^2 + dy^2 + dz^2$ of $\mathbb{R}^3$. This is easy since we already have, $x(t,\theta)$, $y(t,\theta)$, $z(t,\theta)$. We find,

$$g = (\dot{r}^2 + \dot{z}^2)dt^2 + r^2 d\theta^2.$$

If we parametrize the curve by arc length i.e $\dot{r}^2 + \dot{z}^2 = 1 \implies |\dot{r}| < 1$,

$$g = dt^2 + r^2 d\theta^2.$$

We can define a broader class of metric of above type on a manifold $I \times S^1$, using the frame $\partial_t, \partial_\theta$ and coframe $dt, d\theta$,

$$g = \eta^2(t)dt^2 + \rho^2(t)d\theta^2,$$

which do not necessarily arise from surface of revolution.

We can generalize this to metrics on $I \times S^{n-1}$ of type

$$g = dt^2 + \rho^2(t)ds^2_{n-1}$$

where ds^2_{n-1} is the canonical metric on S^{n-1}. Smoothness of the metric at the origin of $\mathbb{R}^n$ for cases where $\rho(0) = 0$ however needs futher analysis, and conditions on $\rho(t)$ can be derived.

Such metrics are called *rotationally symmetric*.

Fundamental examples for such metrics can be realised over the manifolds which we discussed above with $O(n) \subset Iso$.

Example 2.2.4 (Hyperbolic, Euclidean, Spherical spaces in one sweep)**.** $S^2(R)$ is a surface of revolution of the following curve (circle),

$$t \mapsto R\left(\sin\left(\frac{t}{R}\right), 0, \cos\left(\frac{t}{\mathbb{R}}\right)\right).$$

So the metric induced is,

$$dt^2 + R^2 \sin^2\left(\frac{t}{R}\right) d\theta^2.$$

By letting $R \to iR$, we can also rewrite this as

$$dt^2 + R^2 \sinh^2(\frac{t}{R}) d\theta^2.$$

which precisely comes from the revolution of the curve

$$t \mapsto R\left(\sinh\left(\frac{t}{R}\right), 0, \cosh\left(\frac{t}{R}\right)\right)$$

induced by the standard metric on $\mathbb{R}^{2,1}$.

We can identify the functions $\rho(t)$ in the above cases with the unique solution to,

$$\begin{aligned} \ddot{x}(t) + x(t) &= 0 \\ x(0) &= 0 \\ \dot{x}(0) &= 1. \end{aligned}$$

depending on the sign of k. Denote the solutions as sn_k.

We thus write the 1-parameter family of metrics

$$dt^2 + sn_k^2(t) d\theta^2$$

which corresponds to the following rotationally symmetric spaces depending on the value of k

$$I \times S^1 \overset{isometry}{=} \begin{cases} S^2\left(\frac{1}{\sqrt{k}}\right) & k > 0 \\ \mathbb{R}^2 & k = 0 \\ H^2\left(\frac{1}{\sqrt{-k}}\right) & k < 0. \end{cases}$$

This holds true even when we consider the generalized metrics of type,

$$dt^2 + sn_k^2(t)ds_{n-1}^2$$

on $I \times S^{n-1}$.

2.3 Musical Isomorphisms

Using the metric, one can define isomorphisms between tensors of ranks (s, t) and $(s-k, t+k)$ where $s-k > 0, t+k > 0$. This is essentially because $TM \cong T^*M$ as Riemannian manifolds (also). A vector $v \in TM$ can be mapped to $g(v, .) \in T^*M$.

Suppose $\{E_i\}$ is a frame of TM and $\{\sigma_i\}$ its coframe.

Let's find the covector corresponding to E_i defined by the above isomorphism. Call it E_i^*.

$$E_i^*(v) = g\left(E_i, v)\right) = g\left(E_i, E_j\right)\sigma^j(v).$$

So, $E_i \mapsto g_{ij}\sigma^j$.

And similarly, let σ^i be mapped to vec^i, such that $g\left(vec^i, v\right) = \sigma^i(v)$ for all vector fields v of TM. In particular,

$$vec^{ij} g\left(E_j, E_k\right) = \delta_k^i i$$
$$vec^{ij} = g^{ij}.$$

where we have denoted the inverse of g with upper indices.

So, $\sigma^i \to g^{ij}E_j$.

This way we can *raise* and *lower* indices of any (s, t) tensor in $\mathcal{T}^{(s,t)}(M)$ which can also be uniquely written as a $(s-k, t-k)$ tensor for each k. Let's see this formally through examples.

Example 2.3.1. Consider a (1,1) tensor $Ric = Ric^i_j E_i \otimes \sigma^j$. This can equally be represented by a (0,2) tensor,

$$Ric = Ric_{ij}\sigma^i \otimes \sigma^j.$$

where Ric_{ij} is obtained by mapping E_i, E_j to its covectors. That is, $Ric_{jk} = Ric^i_j g_{ik}$. We can also represent it as a $(2,0)$ tensor as,

$$\begin{aligned} Ric &= Ric^{ij} E_i \otimes E_j \\ &= Ric^i_k g^{kj} E_i \otimes E_j. \end{aligned}$$

Example 2.3.2. Consider a (1,3) tensor $R = R^i{}_{jkl} E_i \otimes \sigma^j \otimes \sigma^k \otimes \sigma^l$. This has following equivalent representations,

As (0,4) tensor,

$$\begin{aligned} R &= R_{mjkl}\sigma^m \otimes \sigma^j \otimes \sigma^k \otimes \sigma^l \\ &= R^i{}_{jkl} g_{im}\sigma^m \otimes \sigma^j \otimes \sigma^k \otimes \sigma^l. \end{aligned}$$

As a (2,2) tensor,

$$\begin{aligned} R &= R^{im}_{kl} E_i \otimes E_m \otimes \sigma^k \otimes \sigma^l \\ &= R^i{}_{jkl} g^{jm} E_i \otimes E_m \otimes \sigma^k \otimes \sigma^l. \end{aligned}$$

One could have raised the other indices instad as well which defines another (2,2) tensor representation,

$$
\begin{aligned}
R &= R^{im}_{jl} E_i \otimes \sigma^j \otimes E_m \otimes \sigma^l \\
&= R^{i}_{jkl} g^{km} E_i \otimes \sigma^j \otimes E_m \otimes \sigma^l .
\end{aligned}
$$

But all such (2,2) representations are same in the sense that they eat two vectors and two covectors.

To summarise, the metric allows us to define the isomorphisms $\mathcal{T}^{(s,t)}(\mathcal{M}) \cong \mathcal{T}^{(s-k,t-k)}(\mathcal{M})$. And further, there is an ordering difficulty for any tensor $T^{(s,t)}(\mathcal{M})$. We can be more careful by using a *positional* notation.

In the above example, the two (2,2) tensor representations can be written more carefully as,

$$
\begin{aligned}
R &= R^{im}{}_{kl} E_i \otimes E_m \otimes \sigma^k \otimes \sigma^l \\
R &= R^{i}{}_{j}{}^{m}{}_{l} E_i \otimes \sigma^j \otimes E_m \otimes \sigma^l .
\end{aligned}
$$

Contractions.

We can also *contract* indices by taking the trace of tensors.

For a (1,1) tensor $T = T^i_j E_i \otimes \sigma^j$ this is just $C(T) = T^i_i$. For the (0,2) or (2,0) representation of T, we can convert this to (1,1) representation first and then take trace, $C(T) = T_{ij} g^{ij} = T^{ij} g_{ij} = T^i_i$. These equalities immediately follow from the relations between T^i_j, T^{ij}, T_{ij}.

Inner product of tensors.

The Euclidean norm can be defined by, $|T| = tr\,(T \circ T^*)$. T^* is just a reordering of components of T which is clear in the positional notation. It is basically dual to the multilinear map T (or the adjoint).

2.4 Derivatives

2.4.1 Gradient

Definition 2.4.1. *Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Then we define the gradient vector field as $grad_f = \nabla f$ satisfying $g\left(grad_f, v\right) = df(v)$ for all $v \in TM$.That is, $grad_f$ is the covariant version of df,*

$$df \mapsto grad_f$$

mapped under the isomorphism between $TM andT^*M$.

Remark. Note that the above isomorphism is metric dependent here. Any vector space V and V^* are isomorphic but there is no canonical map - it depends on the basis we choose over V. However if V is equipped with an inner product (here g on TM), we can define a canonical map that maps vectors to covectors. So, gradient vector fields are well defined (i.e. invariantly) precisely because of the Riemannian structure. For example, if one defines $grad_f = \partial_i(f)\partial_i$ where we know that $df = \partial_i(f)dx^i$ works only as a caratesian definition. In spherical polar coordinates, we know that the form looks different.

The metric thus gives an invariant definition of gradient vector field as follows,

$$grad_f = \nabla f = g^{ij}\partial_i(f)\partial_j$$

2.4.2 Lie Derivative

The idea of lie derivative is to keep track of *change* (of maps) *along* a vector field. Where *along* is determined by the flow generated by the vector field. It is the first-order term in their corresponding taylor expansion on the flow.

Consider a vector field $X \in \mathfrak{X}(M)$ and the local flow $F^t(x)$ generated by it.

For functions $f \in C^\infty(M)$ the Lie derivative $L_X f$ is given by,

$$f(F^t(p)) = f(p) + t(L_X f)(p) + O(t^2)$$

i.e.

$$L_X f(p) = \lim_{t \to 0} \frac{f\left(F^t(p)\right) - f(p)}{t}.$$

This is basically $\frac{d}{dt}\left(f \circ F^t\right) = \left(f \circ F^t\right)' = (F^t)'(f)$ which is precisely the directional derivative of f, $X(f)$ or $df(X)$,

$$L_X f = D_X f = X(f).$$

Lie derivative is essentially a generalization of this operation to other objects on manifolds. As we will see this operation plays an important role.

Consider another vector field Y, the question we ask is how does $Y|_p$ compare with its value obtained via a pull-back due to the flow starting at p due to X. Here, by pull-back we mean the pushforward $F_*^{-t}(Y|_{F(t)}) \in T_p M$. That is,

$$F_*^{-t}(Y|_{F^t(p)}) = Y|_p + t\,(L_X Y)\,|_p + O(t^2)$$

or,

$$L_X Y|_p - \lim_{t \to 0} \frac{F_*^{-t}(Y|_{F^t(p)}) - Y|_p}{t}.$$

This infact turns out to be the Lie Bracket of the vector fields X and Y!

Proposition 2.4.1. *If X, Y are vector fields on M, then $L_X Y = [X, Y]$.*

Proof. Evaluate RHS of $L_X Y|_p$ on $f_p \in C_p^\infty(M)$,

$$\begin{aligned}
\left(F_*^{-t}\left(Y|_F^t(p)\right) - Y|_p\right)(f_p) &= F_{F^t(p)}^{-t}\left(Y|_{F^t(p)}\right)(f_p) - Y|_p(f_p) \\
&= Y_{F^t(p)}\left(\left(f \circ F^{-t}\right)_{F^t(p)}\right) - Y|_p(f_p) \\
&= Y\left(f \circ F^{-t}\right) \circ F^t - Y(f) \\
&= Y\left(f \circ F^{-t}\right) - tX\left(Y\left(f \circ F^{-t}\right)\right) + O(t^2) - Y(f) \\
&= Yf - tY(X(f)) + tX(Y(f)) - Y(f) + O(t^2) \\
&= t[X,Y]f.
\end{aligned}$$

Thus, $L_X Y = [X,Y]$. □

We can also take Lie Derivative of a (0,k) tensor T as follows,

$$F^{t*}T = T + t\left(L_X\right)Y + O(t^2).$$

That is,

$$\left(L_X T\right)\left(Y_1, \ldots, Y_k\right) = \lim_{t \to 0} \frac{F^{t*}T - T}{t}.$$

Evaluating RHS on vector field Y one can easily show the following formula of Lie Derivative in this case.

Proposition 2.4.2. *If X is a vector feld and T a $(0,k)$-tensor on M, then*

$$\left(L_X T\right)\left(Y_1, \ldots, Y_k\right) = D_X\left(T\left(Y_1, \ldots, Y_k\right)\right) - \sum_{i=1}^{k} T\left(Y_1, \ldots, L_X Y_i, \ldots, Y_k\right).$$

L_X, like in case of functions and vector fields acting as derivation, obeys product rule when acted on $(0,k)$-tensors.

Proposition 2.4.3. *If T_1, T_2 are two $(0,k)$ tensors, then*

$$L_X(T_1 \otimes T_2) = \left(L_X T_1\right) \otimes T_2 + T_1 \otimes \left(L_X T_2\right)$$

If $X_p = 0$ at some point p, then $F^t(p) = p$ for all t. So,

$$\begin{aligned} L_X Y|_p &= \lim_{t \to 0} \frac{F_*^{-t}(Y|_p) - Y|_p}{t} \\ &= \frac{d}{dt}\left(F_*^{-t}\right)\Big|_{t=0}\left(Y|_p\right). \end{aligned}$$

$L_X = \frac{d}{dt} F_*^{-t}|_{t=0}$ whenever $X_p = 0$. And for 1-forms, $L_X \theta = \theta \circ L_X$ whenever $X_p = 0$, which only depends on the value of θ at p. The same is true of $(0, k)$-tensors.

Lie Derivatives can also be defined for other general tensors and maps similarly. In particular, we can take Lie Derivative of the Lie Bracket vector field and on a Lie derivative $L_Y T$ define the following on any tensor T,

$$(L_X L)_Y\, T = L_X\left(L_Y T\right) - L_{L_X Y} T - L_Y\left(L_X T\right) = [L_X, L_Y]T - L_{[X,Y]}T.$$

This infact vanishes on all tensors.

Proposition 2.4.4 (Generalized Jacobi Identity)**.** *For all vector fields X, Y and tensors T*

$$(L_X L)_Y\, T = 0.$$

When acted on a vector field Z. This reduces to the usual Jacobi identity satisfied by the Lie Bracket X, Y, Z.

Connection to Exterior Derivative.

Note that we have a definition of exterior derivative on $w \in \Omega^k(M)$ through the formula,

$$dw\left(X_1,\ldots,X_{k+1}\right)=\sum_{i=1}^{k+1}(-1)^i X_i w\left(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}\right)$$
$$+\sum_{i,j}^{k+1}(-1)^{i+j}\, w\left([X_i,X_j],X_1,\ldots,\hat{X}_j,\ldots,X_{k+1}\right)$$

This can be written in terms of the Lie derivative as,

$$d\omega(X_0,X_1,\ldots,X_k)=\frac{1}{2}\sum_{i=0}^{k}(-1)^i(L_{X_i}\omega)(X_0,\ldots,\hat{X}_i,\ldots,X_k)$$
$$+\frac{1}{2}\sum_{i=0}^{k}(-1)^i L_{X_i}\left(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)\right)$$

In particular, we have on 1-form

$$dw\left(X,Y\right)=D_X\left(w(Y)\right)-D_Y\left(w(X)\right)-w\left([X,Y]\right).$$

The following definitions are worth noting.

Definition 2.4.2 (Divergence, Laplacian and Hessian)**.** *Let (M,g) be a Riemannian Manifold.*

1. *The divergence of a vector field X is defined by its action on the volume form - it descsribees the change in the volume form along the flow due to X, i.e. $L_X vol = (divX)vol$.*
2. *The Laplacian of a function $\Delta f = div\nabla f$.*
3. *The Hessian of a function is a $(0,2)$ tensor,*

$$Hessf(X,Y)=\frac{1}{2}\left(L_{\nabla_f g}\right)(X,Y).$$

The above defintion of Hessian is indeed a generalization from that on Euclidean spaces. The Laplacian is the trace of the Hessian and on $\mathbb{R}^n$, the Hessian reduces to it's familiar form.

Let $E_1, \ldots, E_n$ be a positively oriented orthonormal frame. Then,

$$\begin{aligned}
divX &= (L_X vol)(E_1, \ldots, E_n) \\
&= L_X (vol(E_1, \ldots, E_n)) - \sum vol(E_1, \ldots, L_X E_i, \ldots, E_n) \\
&= -\sum g(L_X E_i, E_i) \\
&= \frac{1}{2} \sum \left(L_X \left(g(E_i, E_i)\right)\right) - g(L_X E_i, E_i) - g(E_i, L_X E_i) \\
&= \sum \frac{1}{2} \left(L_X g\right)(E_i, E_i).
\end{aligned}$$

In writing above, we have used the behavior of volume form and that g is symmetric, $g(E_i, E_i) = Vol(E_1, \ldots, E_n) = 1$.

For a Euclidean space with standard metric, consider $f : \mathbb{R}^n \to \mathbb{R}$. Then,

$$\begin{aligned}
L_{\nabla f}\left(\delta_{ij} dx^i dx^j\right) &= L_{\sum \partial_j f \partial_j} \sum dx^i dx^i \\
&= \sum L_{\partial_j f \partial_j} dx^i dx^i \\
&= \sum \left(L_{\partial_j f \partial_j} dx^i\right) dx^i + \sum dx_i \left(L_{\partial_j f \partial_j} dx^i\right) \\
&= \sum \partial_j f \left(L_{\partial_j} dx^i\right) + \sum \partial_j f\, dx^i \left(L_{\partial_j} dx^i\right) \\
&= 2 \sum d\left(\partial_j f\right) dx^i \\
&= 2 \sum \partial_{ji} f\, dx^j dx^i \\
&= 2 Hessf.
\end{aligned}$$

2.5 Connection

In order to generalize the notion of constant or *parallel* vector fields, like in $\mathbb{R}^n$ we introduce the notion of covariant derivative. These are not just vector fields $X = X^i \partial_i$ with constant functions X^i.

There are two ways we can keep track of change in a vector field, it could be a gradient of a function, which is captured by the 2-form $d\theta_X$ which is the exterior derivative of the dual 1-form of X. That is,$\theta_X(Y) = g(X,Y)$ and $X = \nabla f \implies d\theta_X = 0$!. Further, we can also ask how the metric changes along the flow due to X, which is captured by the Lie derivative $L_X g$. It's interesting to note the relevance of Hessian here. A vector field whose Lie Derivative vanishes is called a *Killing field*.

Lemma 2.5.1. $L_X g = 0 \iff$ *the flow F^t generated by X are isometries.*

Consider $\mathbb{R}^2$ with the standard metric. If a vector field X is gradient ∇f , then $d\theta_X = 0$ and the change in metric due to X is essentially $2Hessf$. So, X is killing if and only if $\partial_i X^j + \partial_j X^i = 0$. This gives us the familiar result,

$$X^i = \alpha^i_j x^i + \beta^i; \quad \alpha^i_j = -\alpha^j_i.$$

In $\mathbb{R}^n$ a vector field is *constant* if and only if it is both killing and a gradient field.

To generalize this we have the following tensorial result.

Proposition 2.5.2. *The covariant derivative in $\mathbb{R}^n$ is given by the implicit formula:*

$$2g\left(\nabla_Y X, Z\right) = \left(L_X g\right)(Y,Z) + \left(d\theta_X\right)(Y,Z).$$

Until now we have an idea of $\nabla_Y X$ only in cartesian coordinates, but the above tensorial expression will be used to define the covariant derivative on arbitrary manifolds.

Proof. It suffices to consider caratesian form of $\nabla_Y X$, as $\partial_k X^i)\partial_i$ and commpute the right hand side. □

What we are really interested in is the derivation of a vector field dueto another vector field, depending linearly on it. Such an object is called an *affine connection.*

Theorem 2.5.3 (The Fundamental Theorem of Riemannian Geometry)**.** *The assignment $X \mapsto \nabla X$ on (M,g) is uniquely defined by the following properties:*

1. *$\nabla X : Y \mapsto \nabla_Y X$ is a $(1,1)$ tensor. That is,*

$$\nabla_{\alpha v+\beta w} X = \alpha \nabla_v X + \beta \nabla_w X.$$

2. *$X \mapsto \nabla_Y X$ is a derivation:*

$$\nabla_Y (X_1 + X_2) = \nabla_Y X_1 + \nabla_Y X_2,$$
$$\nabla_Y (fX) = \left(D_Y f\right) X + f \nabla_Y X..$$

   for functions $f : M \to \mathbb{R}$.

3. *Covariant differentiation is torsion free:*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

4. *Covariant differentiation is compatible with metric:*

$$D_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

The assignment satisfying (1) and (2) is called an affine connection. Further (3) and (4) define a uniqe such connection called, *Riemanniann* or *Levi-Civita* connection.

Proof. We use the implicit definition of the covariant derivative to prove this theorem, and the uniqueness result ensures there is no loss of generality. □

We will see the significance of Torsion free property and the metric compatibility later. The following two lemmas describe the dependence of $\nabla_Y X|_p$ on X. Note that the dependence on Y is tensorial i.e linear in Y.

Lemma 2.5.4. *Let M be a manifold and ∇ an affine connection on M.If $p \in M$, $v \in T_p M$, and X, Y are vector fields on M such that $X = Y$ in a neighbordhood $U \ni p$,then $\nabla_v X = \nabla_v Y$.*

Proof. Let $f : M \to \mathbb{R}$ be a bump function vanishing in $M \backslash U$ and $f \equiv 1$ in a neighborhood $V \subset U$ of p. So we have $fX = fY$ on all of M. Thus, at p:

$$\nabla_v fX = f(p)\nabla_v X + D_v(f)X(p) = \nabla_v X.$$

So, $\nabla_v X = \nabla_v fX = \nabla_v fY = \nabla_v Y$. □

As expeected from a derivation, the covariant derivative of X is a local property, and it suffices to have a vector field defined on an open set around a point for it to make sense. Infact something more stronger is true, it suffices for the vector field to be defined along a curve.

Lemma 2.5.5. *Let M be a manifold and ∇ an affine connection on M. If X is a vector field on M and $c : I \to M$ a smooth curve with $\dot{c}(0) = v \in T_pM$, then $\nabla_v X$ depends only on the values of X along c, i.e., if $X \circ c = Y \circ c$, then $\nabla_{\dot{c}} X = \nabla_{\dot{c}} Y$.*

Proof. Let $E_1, \ldots, E_n$ be a frame in the neighborhood of p, andX^i, Y^i the components of vector fields which agree on the curve c. Then,

$$\begin{aligned}
\nabla_v Y &= \nabla_v \left(Y^i E_i\right) \\
&= Y^i(p)\nabla_v(E_i) + D_v Y^i|_p E_i(p) \\
&= X^i(p)\nabla_v(E_i) + D_v X^i|_p E_i(p) \\
&= \nabla_v X.
\end{aligned}$$

□

From the above two lemmas, it is clear that it is possible to compute ∇ in local frames. We will do that soon.

From the uniqueness of the Levi-Civita connection, we can evaluate its behaviour under isometries.

Proposition 2.5.6 (Naturality of the Levi-Civita Connection)**.** *Suppose* (M, g) *and* $(\tilde{M}, \tilde{g})$ *are Riemannian or pseudo-Riemannian manifolds with or without boundary, and let* ∇ *denote the Levi-Civita connection of* g *and* $\tilde{\nabla}$ *that of* $\tilde{g}$*. If* $\varphi : M \to \tilde{M}$ *is an isometry, then* $\varphi^* \tilde{\nabla} = \nabla$*.*

2.5.1 Covariant Derivative of Tensors

The generalization of covariant derivative to general tensors will be important to derive useful formale for Hessian, Laplacian, divergence andalong with giving using tools to probe curvature (tensors).

The covariant derivative takes a (1,0)-tensor X to a (1,1)-tensor ∇X. Similarly we let (s,t)-tensor S go to a (s,t+1)-tensor ∇S. We do this in a way which respects the product rule:

$$\begin{aligned}\nabla S(X, Y_1, \ldots, Y_t) &= (\nabla_X S)(Y_1, \ldots, Y_t) \\ &= \nabla_X (S(Y_1, \ldots, Y_t)) - \sum_{i=1}^{t} S(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_t).\end{aligned}$$

Sending the second term of RHS to the LHS, we see the product rule. The definition for $s \geq 1$ is similar.

Definition 2.5.1. *A tensor S is called parallel, if* $\nabla S = 0$.

In cartesian coordinates of Euclidean spaces, parallel tensors are precisely the constant coefficient tensors. In particular, $\nabla X \equiv 0$ for constant vector fields (irrespective of the coordinates).

Proposition 2.5.7. *On a Riemannian n-manifold* (M, g)

$$\nabla g = 0$$

$$\nabla vol = 0.$$

Proof. It is the metric compatibility of the covariant differentiation that directly ensures g is parallel. Evaluating $\nabla_X vol$ on an orthonormal frame $E_1, \ldots, E_n$ and using the metric compatibility immediately gives that vol is parallel too. □

The covariant derivative of the 1-form df gives precisely the Hessian!

Proposition 2.5.8. *If $f : (M, g) \to \mathbb{R}$, then*

$$\left(\nabla_X df\right)(Y) = g\left(\nabla_X \nabla f, Y\right) = Hessf(X, Y).$$

Definition 2.5.2. *The adjoing to the covariant derivative on (s,t)-tensors is defined as*

$$\left(\nabla^* S\right)\left(X_2, \ldots, X_t\right) = -\sum \left(\nabla_{E^i} S\right)\left(E_i, X_2, \ldots, X_t\right),$$

where $E_1, \ldots, E_n$ is a orthonormal frame. That is, the adjoint covariant derivative gives out a (s,t-1) tensor!

Proposition 2.5.9. *If X is a vector field and θ_X the corresponding 1-form, then*

$$divX = -\nabla^* \theta_X$$

The exterior derivative can now be written simply as,

$$dw\left(X_0, \ldots, X_k\right) = \sum (-1)^i \left(\nabla_{X_i} w\right)\left(X_0, \ldots, \hat{X}_i, \ldots, X_k\right).$$

We can take second covariant derivative of a (s,t)-tensor, $\nabla^2 S$ as follows -

$$\begin{aligned}
\left(\nabla_{X_1, X_2} S\right)\left(Y_1, \ldots, Y_t\right) &= \left(\nabla_{X_1}(\nabla S)\right)\left(X_2, Y_1, \ldots, Y_t\right) \\
&= \left(\nabla_{X_1}\left(\nabla_{X_2} S\right)\right)\left(Y_1, \ldots, Y_t\right) - \left(\nabla_{\nabla_{X_1} X_2}\right)\left(Y_1, \ldots, Y_t\right).
\end{aligned}$$

The second covariant derivative is not symmetric in the arguments and is a hallmark of Riemannian geometry.

We can also take covariant derivative of a covarariant derivative as,

$$(\nabla_X \nabla)_Y T = \nabla_X (\nabla_Y T) - \nabla_{\nabla_X Y} T - \nabla_Y (\nabla_X T).$$

The two covariant derivatives are related to each other via,

$$\nabla^2_{X,Y} T = (\nabla_X \nabla)_Y T + \nabla_Y (\nabla_X T).$$

2.5.2 Connection in local form

In local coordinates,

$$\begin{aligned}
\nabla_Y X &= \nabla_{Y^i \partial_i} X^j \partial_j \\
&= Y^i \nabla_{\partial_i} X^j \partial_j \\
&= Y^i (\partial_i X^i) \partial_j + Y^i X^j \nabla_{\partial_i} \partial_j \\
&= Y^i (\partial_i X^j) \partial_j + Y^i X^j \Gamma^k_{ij} \partial_k,
\end{aligned}$$

where we expanded $\nabla_{\partial_i} \partial_j$ in local coordinates. We see that for different coordinate systems and general Riemannian manifolds, there is an extra correction term. We can find these coefficients in terms of the metric using the implicit definition.

$$\Gamma^k_{ij} = \frac{1}{2} g^{lk} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji} \right) = \frac{1}{2} g^{kl} \Gamma_{ij,l} = \frac{1}{2} g^{kl} g \left(\nabla_{\partial_i} \partial_j, \partial_l \right).$$

The symbols, $\Gamma_{ij,k}$ are called the *Christoffel symbols of first kind* and Γ^k_{ij} are called the *Christoffel symbols of second kind*. Note that these area not tensors! Indeed as wealready know,it is always possible to find coordinates such that $\Gamma = 0$ (but not in others). These are the *locally flat coordinates*.

Normal Coordinates around a point $p \in M$:

$$
\begin{aligned}
g_{ij}|_p &= \delta_{ij}, \\
\partial_k g_{ij}|_p &= 0.
\end{aligned}
$$

Locally Flat Coordinates:

$$
\begin{aligned}
g_{ij}|_p &= \delta_{ij}, \\
\Gamma^k_{ij}|_p &= 0.
\end{aligned}
$$

It turns out Normal Coordinates always exists and thus so does Locally flat coordinates. The covariant derivative is then calculated easily.

The defining properties of a Riemannanian connection tell us the following about the connection coefficients Γ.

Christoffel symbols Γ^k_{ij} are symmetric in ij. Since, ∇ is torsion free and $[\partial_i, \partial_j] = 0$

$$
\begin{aligned}
\Gamma^k_{ij}\partial_k &= \nabla_{\partial_i}\partial_j \\
&= \nabla_{\partial_j}\partial_i \\
&= \Gamma^k_{ji}.
\end{aligned}
$$

Chirstoffel symbols completely determine the first derivatives of the metric. Metric compatibility of connection gives,

$$
\begin{aligned}
\partial_k g_{ij} &= g\left(\nabla_{\partial_k}\partial_i, \partial_j\right) + g\left(\partial_i, \nabla_{\partial_k}\partial_j\right) \\
&= \Gamma_{ki,j} + \Gamma_{kj,i}.
\end{aligned}
$$

For higher tensors say, (1,k) tensor S:

$$
\begin{aligned}
S &= S^i_{j_1,\ldots,j_k} E_i \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_k}, \\
\nabla S &= \nabla S^i_{j_1,\ldots,j_{k+1}} E_i \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_k} \otimes \sigma^{j_{k+1}}..
\end{aligned}
$$

We obtain the coefficients $\nabla_{j_0} S^i_{j_1,\ldots,j_k} \equiv (\nabla S)^i_{j_0,\ldots,j_{k+1}}$ through $\nabla_{j_0} S \equiv \nabla_{E_{j_0}} S$.

2.6 Curvature

We are now ready to define the curvature of a Riemannian Manifold. Curvature enters the theory of Riemannian geometry as a local invariant which are preserved under local equivalances between two structures under study, here the structures being Riemannian Manifolds.

The curvature of the space can be probed by parallely transporting a vector field along suitable directions. We wish to compute the difference in the vector field at a point after this transport. In $\mathbb{R}^2$ for example, take a vector field Z and transport it parallely along x-axis, then along all the coordinate lines parallel to y-axis. We wish to know if $\nabla_{\partial_1} Z = 0$ i.e. if Z is parallel along the x-axis. Infact since $\nabla_{\partial_1} Z = 0$ on x-axis, showing $\nabla_{\partial_2} \nabla_{\partial_1} Z = 0$ suffices by uniqueness. And since Z is parallel along y-axis, showing $\nabla_{\partial_2} \nabla_{\partial_1} Z = \nabla_{\partial_1} \nabla_{\partial_2} Z$ then suffices. Ofcourse, this is exactly what happens in Euclidean spaces.

For any vector fields X, Y, Z:

$$\nabla_X \nabla_Y Z = \nabla_X \left(Y(Z^k) \partial_k \right) = XY(Z^k) \partial_k$$

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

In particular on $\mathbb{R}^2$, if we take $X = \partial_1, Y = \partial_2$ - we indeed have, $\nabla_1 Z = 0$!

Definition 2.6.1. *A Riemannian manifold is said to be flat if it is localy isometryic to a Euclidean space, that is, if every point has a neighborhood that is isometryic to an open set in $\mathbb{R}^n$ with its Euclidean metric.*

Definition 2.6.2. *A connection ∇ on a smooth manifold M is said to satisfy the flatness criterion if whenever X, Y, Z are smooth vector fields defined on an open subset of M, the following identity holds:*

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z$$

Since the Euclidean connection satisfies the flatness criterion and by the naturality of connection due to local isometries, we have the following neat result.

Proposition 2.6.1. *If (M, g) is a flat Riemannian manifold, then its Levi-Civita connection satisfies the flatness criterion.*

In what follows we develop the tools required to approach the other direction, i.e to find sufficient conditions for a manifold to be flat.

2.6.1 Curvataure Tensor

Proposition 2.6.2. *On a Riemannian manifold (M, g) we define the (1,3) curvature tensor via the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$:*

$$
\begin{aligned}
R(X,Y)Z &= \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.
\end{aligned}
$$

Proof. Note that covariant derivatives are tensorial in X, Y. Using the product rule of covariant derivatives it is easy to also show $R(X,Y)fZ = fR(X,Y)Z$. □

Proposition 2.6.3. *The Riemannian curvature tensor $R\,(X, Y, Z, W)$ satisfies the following properties:*

1. *R is skew-symmetric in the first two and last two entries:*

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z).$$

2. *R is symmetric between the first two and last two entries:*

$$R(X,Y,Z,W) = R(Z,W,X,Y).$$

3. *R satisfies a cyclic permutation property called Bianchi's first identity:*

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

4. ∇R *satisfies a cyclic permutation property called ianci's second identity:*

$$(\nabla_Z R)_{X,Y} W + (\nabla_X R)_{Y,Z} W + (\nabla_Y R)_{Z,X} W = 0.$$

In local coordinates, we write:

$$Rm = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where $R\left(\partial_i, \partial_j\right)\partial_k = R^l_{ijk}\partial_l$. We can compute these functions using the connection coeficients:

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

We can define the (0,4) form of the curvataure tensor often called the *Riemann Curvature tensor* as,

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

Proposition 2.6.4. *The curvature tensor is a local isometry invariant: If* (M, g) *and* $(\tilde{M}, \tilde{g})$ *are Riemannian manifolds and* $\phi : M \to M$ *is a local isometry, then* $\phi^* \widetilde{Rm} = Rm$.

Proof. This again follows from the naturality of connection. □

2.6.2 Flat Manifolds and the Curvature Tensor

We will make the function of the curvature tensor as an obstruction to Euclidean space precise here.

Lemma 2.6.5. *Suppose* M *is a smooth manifold and* ∇ *is any connection on* M *satisfying the flatness citerion. Given* $p \in M$ *and any vector* $v \in T_pM$*, there exists a parallel vector field* V *on a neighborhood of* p *such that* $V_p = v$.

Proof. The idea is to consider a cubic coordinate neighborhood around a point p, and parallel transport the vector field V along x^1-axis, then successively along $x^2, \ldots, x^n$ at every point of the x^1-axis.We want to show that $\nabla_{\partial_i} V = 0$ for all $i = 1, \ldots, n$. We can do this by induction. This is just a generalization of the procedure which we used to motivate the flateness criterion. Since it is given here that the flatness criterion is satisfied, the result follows. □

This allows us to now give a sufficient condition for a manifold to be flat.

Theorem 2.6.6. *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proof. The forward direction was already established. Suppose the curvature tensor vanishes. Then ∇ satisfies the flatness criterion. We use the previous result to construct a parallel orthonormal frame in a neighborhood of each point which is the important characteristic of a Euclidean space. Startin from an orthonormal basis $(b_1, \ldots, b_n)$ on T_pM, we can construct an parallel orthonormal frame in a neighborhood of p. Because these are parallel and the connection is flat, they commute. It then follows that there exists coordinates $(y^1, \ldots, y^n)$ such that $E_i = \partial_i$ for $i = 1, \ldots, n$and $g_{ij} = \pm\delta_{ij}$. This defines a local isomtery to the Euclidean space. □

In particular, the curvature tensor is related to the change in vector field due to a series of parallel transports as follows.

Theorem 2.6.7. *Let (M, g) be a Riemannian manifold; let I be an open interval containing* 0*; let* : $\Gamma : I \times I \to M$ *be a smooth one-parameter familyof curves; and let* $p = \Gamma(0,0)$, $x = \partial_s\Gamma(0,0)$,*and* $y = \partial_1\Gamma(0,0)$. *For any* $s_1, s_2, t_1, t_2 \in I$, *let* $P^{s_1,t_2}_{s_1,t_1} : T_{\Gamma(s_1,t_1)}M \to T_{\Gamma(s_1,t_2)}M$ *denote parallel transport along the curve* $t \mapsto \Gamma(s_1, t_1)$ *from time* t_1 *to time* t_2*, and let* $P^{s_2,t_2}_{s_1,t_1}$: $T_{\Gamma(s_1,t_1)}M \to T_{\Gamma(s_2,t_1)}M$ *denote the parallel transport along the curve* $s \mapsto \Gamma(s_1, t_1)$ *from time* s_1 *to time* s_2. *Then for every* $z \in T_pM$,

$$R(x, y)z = \lim_{\delta,\epsilon \to 0} \frac{P^{0,0}_{\delta,0} \circ P^{\delta,0}_{\delta,\epsilon} \circ P^{\delta,\epsilon}_{0,\epsilon} \circ P^{0,\epsilon}_{0,0}(z) - z}{\delta\epsilon}$$

2.6.3 Ricci and Scalar Curvatures

As a matter of simplifying the content of the curvature tensor, we define a (0,2) tensor field called *Ricci Curvature* or *Ricci Tensor* as the trace of Rm over the first and last indices,

$$Rc(Y,Z) = tr\,(X \to R(X,Y)Z)\,; \quad Rc = R_{ij}dx^i \otimes dx^j; R_{ij} = R^k_{kij} = g^{kl}R_{kijl}.$$

Note that this is a symmetric 2-tensor.

And define *scalar curvature* as the trace of the Ricci tensor:

$$R = tr_g Rc = R^i_i.$$

It is also useful to define the *traceless Ricci tensor* as,

$$\overset{\circ}{R}c = Rc - \frac{1}{n}Rg.$$

This is infact an orthogonal decomposition.

Proposition 2.6.8 (Contracted Bianchi Identity). *Let (M,g) be a Riemannianmanifold. The covariant derivatives of the Riemann, Ricci, and scalar curvatures of g satisfy the following identities:*

$$tr_g(\nabla Rm) = -D(Rc)$$
$$tr_g\,(\nabla Rc) = \frac{1}{2}dS..$$

with trace over the first and last indices. In components, this is

$$R^l_{ijkl;} = R_{jk;l} - R_{jl;k}$$
$$R^i_{il;} = \frac{1}{2}S_{;l}..$$

Definition 2.6.3. *A Riemannian metric is said to be an Einstein metric if its Ricci tensor is a constant multiple of the metric,*

$$Rc = \lambda g, \quad \lambda \textit{ is constant.}$$

This is the Einstein equation, and the next proposition asserts that for connected manifolds, the physical vacuum Einstein equation is indeed an Einstein equation in the above sense.

Proposition 2.6.9 (Schur's Lemma)**.** *Suppose (M, g) is a connected Riemannian manin-fold of dimension $n \geq 3$whose Ricci tensor satisfies $Rc = fg$ for some smooth real-valued function f. Then f is constant and g is an Einstein metric.*

Proof. Taking the trace of $Rc = fg$, we see that, $\mathring{Rc} = 0 \implies \nabla \mathring{Rc} = 0$. So,

$$R_{ij;k} - \frac{1}{n}_{;k} g_{ij} = 0$$
$$\frac{1}{2} S_{;j} - \frac{1}{n} S_{;j}.$$

For $n \geq 3$, this gives, $S_{;j} = 0$ which is basically $\partial_j S$. Since,M is connected, we have S is constant, and so is f. □

Corollary 2.6.10. *If (M, g) is a connected Riemannian manifold of dimension $n \geq 3$, then g is Einstein if and only if $\mathring{Rc} = 0$.*

2.6.4 Conformal Geometry

We now deal with the information in Riemannian curvature that is not encoded in Ricci curvature. As we will see, this information is central to conformal geometry. It is useful to algebraize the situation of curvature tensor as follows.

Suppose V is an n-dimensional real vector space. Let $\mathcal{R}(V^*) \subseteq T^4(v^*)$ dennote the vector space of all covariant 4-tensors T on V that have the symmetries of the (0,4) Riemann Curvature tensor:

1. $T(x, y, z, w) = -T(y, x, z, w) = T(y, x, w, z)$.

2. $T(x, y, z, w) = T(z, w, x, y)$.

3. $T(x, y, z, w) + T\left(y, z, x, w\right) + T\left(z, x, y, w\right) = 0.$

An element of $\mathcal{R}(V^*)$ is called an *algebraic curvature tensor* on V.

Proposition 2.6.11. *If the vector spaace V has dimension n, then*

$$dim\mathcal{R}(V^*) = \frac{n^2\left(n^2-1\right)}{12}.$$

Proof. Consider the linear subspace $\mathcal{B}(V^*)$ of $T^4(V^*)$ satisfying only (1) and (2). Then the linear map

$$\begin{aligned}\phi : \Sigma^2\left(\Lambda^2(V)^*\right) &\to \mathcal{B}(V^*)\\ B \mapsto \phi(B)(x, y, z, w) &= B\left(x \wedge y, z \wedge w\right)\end{aligned}$$

is an isomorphism, with an inverse given by $\phi^{-1}\left(T\right)\left(b_i \wedge b_j, b_k \wedge b_l\right) = T\left(b_i, b_j, b_k, b_l\right)$.

So,

$$dim\left(\mathcal{B}(V^*)\right) = dim\left(\Sigma^2\left(\Lambda^2(V)^*\right)\right) = \frac{{}^nC_2\left({}^nC_2+1\right)}{2}.$$

And the required $\mathcal{R}(V^*)$ is basically the kernel of the map,

$$\begin{aligned}\pi : \mathcal{B}(V^*) &\to T^4(v^*)\\ T \mapsto \pi(T)(x, y, z, w) &= \frac{1}{3}\left(T(x, y, z, w) + T(y, z, x, w) + T(z, x, y, w)\right).\end{aligned}$$

By the symmetries (1) and (2), $alt(T)$ reduces precisely to the form of the image of π. And since, $T \in \Lambda^4(V^*)$ satisfies (1) and (2), we have $\pi(T) = alt(T) = T$. Thus $\pi(\mathcal{B}(V^*)) = \Lambda^4(V^*)$ and

$$dim\left(\mathcal{R}(V^*)\right) = dim\left(\mathcal{B}(V^*)\right) - dim\left(\Lambda^4(V^*)\right).$$

□

If there is a scalar product defined on V, i.e. $g \in \Sigma^2(V^*)$. then we would like ask if the map $tr_g : \mathcal{R}(V^*) \to \Sigma^2(V^*)$ is surjective.

Definition 2.6.4. *There is natural way to construct an algebraic curvature tensr out of two symmmetric 2-tensors. Given $h, k \in \Sigma^2(V^*)$, define the Kulkani-Nomizu product of h and k $h \owedge k$ as:*

$$h \owedge k\left(x, y, z, w\right) = h\left(x, y\right) k\left(z, w\right) + h\left(y, z\right) k\left(x, w\right) \\ - h\left(x, z\right) k\left(y, w\right) - h\left(y, w\right) k\left(x, z\right) ..$$

Some properties of this product immediately follow.

Lemma 2.6.12 (Properties of the Kulkarni-Nomoizu Product)**.** -

1. *$h \owedge k$ is an algebraic tensor.*

2. *$h \owedge k = k \owedge h$.*

3. *$tr_g\left(h \owedge g\right) = (n-2)h + (tr_g h)$.*

4. *$tr_g\left(g \owedge g\right) = 2(n-1)g$.*

5. *$\langle T, h \owedge g\rangle_g = 4\langle tr_g T, h\rangle_g$, where T is an algebraic curvature tensor on V.*

With this, we can define the inverse map from $\Sigma^2(V^*)$ to $\mathcal{R}(V^*)$.

Proposition 2.6.13. *Let (V, g) be an n-dimensional scalar product space with $n \geq 3$, and define a linear map $G : \Sigma^2(V^*) \to \mathcal{R}(V^*)$ by,*

$$G(h) = \frac{1}{n-2}\left(h - \frac{tr_g h}{2(n-1)} g\right) \owedge g.$$

Then $tr_g(G(S)) = S$ and $Im(G) = ker(tr_g)^\perp$.

Proof. $tr_g(G(S)) = S$ follows easily from the definition. So tr_g is surjective and G is injective. And we already saw that $\langle T, G(h)\rangle = 0$ for $T \in ker(tr_g)$. So $Im(G) = ker(t_g)^\perp$. $\square$

We now finally apply this to the case of Riemannian and Ricci curvature tensors.

Definition 2.6.5 (Weyl Tensor)**.** *Suppose (M, g) is a Riemannian manifold. Define the Schouten tensor of g, denoted by P, to be the following symmetric 2-tensor:*

$$P = \frac{1}{n-2}\left(Rc - \frac{R}{2(n-1)}g\right);$$

and the Weyl tensor of g to be the following algebraic curvature tensor:

$$\begin{aligned} W &= Rm - P \owedge g \\ &= Rm - \frac{1}{n-2}Rc \owedge g + \frac{R}{2(n-1)(n-2)} g \owedge g. \end{aligned}$$

So,$G(Rc) = P \owedge g = G(tr_g Rm)$ The Weyl tensor captures precisely that part of Riemann curvature tensor which the Ricci tensor leaves out! As we will soon see, this infact this also captures (all) the conformal data of the manifold!

Corollary 2.6.14. *For every Riemannian manifold (M, g) of dimension $n \geq 3$, the trace of the Weyl tensor is zero, and $Rm = W + P \owedge g$ is the orthogonal decomposition of Rm corresponding to $\mathcal{R}(V^*) = ker(tr_g) \oplus ker(tr_g)^\perp$.*

Corollary 2.6.15. *Let V be an n-dimensional real vector space.*

1. *If $n = 0$ or $n = 1$, then $\mathcal{R}(V^*) = \{0\}$.*
2. *If $n = 2$, then $\mathcal{R}(V^*)$ is 1-dimensional, spanned by $g \owedge g$.*
3. *If $n = 3$, then $\mathcal{R}(V^*)$ is 6-dimensional and $G : \Sigma^2(V^*) \to \mathcal{R}(V^*)$ is an isomorphism.*

This immediately gives the following:

- In 3 dimensions, the Weyl tensor vannishes identically and the curvature tensor is entirely specified by the Ricci tensor!
- In 2 dimensions, the Riemann and Ricci tensors are completely determined by the scalar curvature as follows:

$$Rm = \frac{1}{4}Rg \owedge g, \quad Rc = \frac{1}{2}Rg.$$

Thus the content of the Riemann Curvature tensor is given by the Weyl and Ricci (Schouten) tensors. And the flatness is given by the whole curvture tensor. We now move to the conformal content of the curvature tensor, that is rather than studying local invariants under local isometries(which preserve distances and thus angles), we would like to probe the more weaker conformal transformations which only preserves angles and study their local invariants. It turns out, Weyl tensor is roughly the conformal content of the manifold.

Definition 2.6.6. *Two metrics g_1, g_2 on a manifold M are said to be conformally related to each other if there is a positive function $f \in C^\infty(M)$ such that $g_2 = f g_1$. Given two Riemannian manifolds (M, g_1) and $(\tilde{M}, \tilde{g}$, a diffeomorphism $\phi : M \to \tilde{M}$ is called a conformal diffeomorphis (or a conformal transformation) if it pulls $\tilde{g}$ bask to a metric that is conformal to g:*

$$\phi^* \tilde{g} = f g \quad \textit{for some positive } f \in C^\infty(M).$$

Two Riemannian manifolds are said to be conformally equivalent if there is a conformal diffeomorphism between them.

Indeed, conformal relation defines an equivalence relation on the collection of Riemannian metric on a manifold.

Definition 2.6.7. *A Riemannian manifold is said to be locally conformally flat if every point of M has a neighborhood that is conformally equivalent to an open set in $\left(\mathbb{R}^n, g_{\mathbb{R}^n}\right)$ with its standard metric.*

Note that $S^n \{B\}$ is conformally equivalent to $\mathbb{R}^n$ with stereographic projection as the conformal diffeomorphism.

We have the following important results on the behaviour of the curvature tensor and its orthogonal elements.

Proposition 2.6.16 (Conformal Transformation of the Levi-Civita Connection)**.** *Let* (M, g) *be a Riemannian or pseudo-Riemannian n-manifold (with or without boundary), and let* $\bar{g} = e^{2f} g$ *be any metric conformal to* g*. If* ∇ *and* $\bar{\nabla}$ *denote the Levi-Civita connections of* g *and* $\bar{g}$*, respectively, then*

$$\bar{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - \langle X, Y\rangle_g \operatorname{grad}_g f.$$

In any local coordinates, the Christoffel symbols of the two connections are related by

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + f_i \delta^k_j + f_j \delta^k_i - g^{kl} f_k g_{ij},$$

where $f_i := \partial_i f$ *is the ith component of* $\nabla f = df = f_i dx^i$*.*

Theorem 2.6.17 (Conformal Transformation of the Curvature)**.** *Let* g *be a Riemannian or pseudo-Riemannian metric on an n-manifold* M *with or without boundary,* $f \in C^\infty(M)$*, and* $\bar{g} = e^{2f} g$*. In the Riemannian case, the curvature tensors of* $\bar{g}$ *(represented with tildes) are related to those of* g *by the following formulas:*

$$\begin{aligned}
\tilde{R}m &= e^{2f}\left(Rm - (\nabla^2 f) \odot g + (df \otimes df) \odot g - \tfrac{1}{2}|df|_g^2 (g \odot g)\right),\\
\tilde{R}c &= Rc - (n-2)(\nabla^2 f) + (n-2)(df \otimes df) - (\Delta f + (n-2)|df|_g^2) g,\\
\tilde{S} &= e^{-2f}\left(S - 2(n-1)\Delta f - (n-1)(n-2)|df|_g^2\right).
\end{aligned}$$

where the curvatures and covariant derivatives on the right are those of g*, and* $\Delta f = \operatorname{div}(\operatorname{grad} f)$*. If in addition* $n \geq 3$*, then*

$$\begin{aligned}
\tilde{P} &= P - \nabla^2 f + (df \otimes df) - \tfrac{1}{2}|df|_g^2 g,\\
\tilde{W} &= e^{2f} W.
\end{aligned}$$

In the pseudo-Riemannian case, the same formulas hold with each occurrence of $|df|_g^2$ *replaced by* $\langle df, df\rangle_g$*.*

Out of all the above formulae, the one that concerns us is the transformation $\tilde{W} = e^{2f} W$. One can derive all the above formula by using local coordinates. And in case of curvatures, we can simplify the computation by going to the locally flat frame. Here's the geometric coroallary of the above result.

Corollary 2.6.18. *Suppose (M, g) is a Riemannian Manifold of dimension $n \geq 3$. If g is locally conformally flat, then its Weyl tensor vanishes identically.*

Proof. Since, there is a local conformal diffeomorphism to $\mathbb{R}^n$, this induces a pull back metric $\tilde{g} = e^{2f} g$ in a neighborhod. But $Rm_{\tilde{g}} = 0$ so, $W_{\tilde{g}} = 0$. By the simple transformation of the Weyl Tensor, we have $W_g = 0$ too in the neighborhood! □

We would like to now study the sufficient condition for the manifold to be conformally flat. Recall that in dimension 3, the Weyl tensor vanishes for *all* manifolds. We have another tensor which captures the conformal data in 3 dimensions.

Definition 2.6.8. *On a Riemannian manifold, define the Cotton 3-tensor as the exterior covariant derivative of the Schouten 2-tensor P:*

$$C(X, Y, Z) = -DP(X, Y, Z) = -\nabla P(X, Y, Z) + \nabla P\,(X, Z, Y)$$

$$C_{ijk} = P_{ij;k} - P_{ik;j}.$$

Proposition 2.6.19. *Suppose (M, g) is a Riemannian manifold of dimension $n \geq 3$, and let W and C denote the Weyl and Cotton tensors, respectively. Then*

$$tr_g\,(\nabla W) = (n-3)\,C,$$

where the trace is on the first and last indices of the 5-tensor ∇W.

Corollary 2.6.20. *Suppose (M, g) is a Riemannian manifold. If $\dim M \geq 4$ and the Weyl tensor vanishes identically, then so does the Cotton tensor.*

Proposition 2.6.21 (Conformal Invariance of Cotton tensor in Dimension 3)**.** *Supppose* (M, g) *is a Riemannian 3-manifold, and* $\tilde{g} = e^{2f} g$ *for some* $f \in C^\infty(M)$*. Then the cotton tensors of both the metrics are equal:* $C_{\tilde{g}} = C_g$*.*

Proof. An long explicit coordinate calculation. □

Corollary 2.6.22. *If* (M, g) *is a locally conformally flat 3-manifold, then the Cotton tensor of* g *vanishes identically.*

Now to the theorem that completes our claim that Weyl tensor (and cotton tensor) hold all the conformal data.

Theorem 2.6.23 (Weyl-Schouten)**.** *Suppose* (M, g) *is a Riemannian manifold of dimension* $n \geq 3$*.*

1. *If* $n \geq 4$*, then* (M, g) *is locally conformally flat if and only if its Weyl tensor is identically zero.*

2. *If* $n = 3$*, then* (M, g) *is locally conformally flat if and only if its Cotton Tensor is identically zero.*

Note that for 1-dimensional Riemannian manifolds, the vector space of algebraic curvatures is zero-dimensional. So all Riemannian 1-manifolds have $Rm = 0$ and are thus flat.

Lemma 2.6.24. *Every Riemannian 2-manifold is locally conformally flat.*

Although not given by the Weyl-Schouten theorem, there are so-called **isothermal coordinates** on all 2-manifolds, which give local conformal equivalences; the proof of which involves PDE theory and Complex analysis.

Chapter 3

Geometry of AdS Spacetime

It is important to understand the symmetry structure (isometries) of the metric space to relate to the conserved quantities of the theory of gravity in that space, and also the causal structure of it. After a general discussion of the isometries and Killing vector fields for general space, with sphere and flat as examples, we define the anti-de Sitter space as the negatively curved maximally symmetric space living inside a higher-dimensional pseudo-Euclidean space of a hyperbolic kind. We then understand this space by giving different possible coordinates on it. We note in particular the hyperbolic and conformally flat nature, along with the notion of conformal boundary of AdS. One can extract some physics in this space by constructing the Penrose diagram - as a way to identify the causal structure. The process of drawing such diagrams is discussed along with examples of Minkowski and AdS spaces. We end by noting that there is not a unique but a arbitrary way to approach the boundary of AdS thus defining a conformal class of metrics on the boundary.

3.1 Isometries, Killing Vector Fields and Maximal Symmetry

In the general theory of relativity, there is a vast freedom in choosing the coordinates. The coordinate transformations which preserve the form of the metric are called isometries. That is $x \longrightarrow x'$ is an isometry if $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$. Infinitesimally we write the transformation as $x'^{\mu} = x^{\mu} + \epsilon\xi^{\mu}(x)$ with ϵ an infinitesimal parameter. Using the tensor transformation law for the metric we have,

$$g_{\rho\sigma}(x') \overset{\text{iso.}}{=} g'_{\rho\sigma}(x') \overset{\text{tens.}}{=} g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \tag{3.1}$$

$$= g_{\mu\nu}(x)\left(\delta^\mu_\rho - \epsilon\partial_\rho\xi^\mu(x)\right)\left(\delta^\nu_\sigma - \epsilon\partial_\sigma\xi^\nu(x)\right) \tag{3.2}$$

$$g_{\rho\sigma}(x) + \epsilon\xi^\lambda\partial_\lambda g_{\rho\sigma} = g_{\rho\sigma}(x) - \epsilon g_{\mu\sigma}\partial_\rho\xi^\mu - \epsilon g_{\rho\nu}\partial_\sigma\xi^\nu \tag{3.3}$$

We thus have what we call the killing equation,

$$g_{\mu\sigma}\partial_\rho\xi^\mu + g_{\rho\nu}\partial_\sigma\xi^\nu + \xi^\lambda\partial_\lambda g_{\mu\nu} = 0 \tag{3.4}$$

and in terms of the covariant derivative, this becomes,

$$\nabla_a\xi_b + \nabla_b\xi_a = 0 \tag{3.5}$$

The ξ^μs satisfying the above equation are called the killing vector fields (KVFs) named after William Killing. This condition allows us to either restrict the metrics (solutions of Einstein Equation) using a set of KVFs or find out the KVFs for a given metric. In what follows, we will probe both possibilities by first finding linearly independent KVFs for familiar spaces and then analysing the form of metric assuming *maximal* number of Killing Vectors, which involves some non-trivial calculations.

Note that any linear combination of killing vectors is also a killing vector, and the number of linearly independent solutions of 1.5 is independent of the coordinates, reflecting the fact that Isometry is an intrinsic property of the space.

3.1.1 Euclidean Space

On a 3-dimensional Euclidean space, we have $g = dx^2 + dy^2 + dz^2$. Thus, the killing condition reads,

$$g_{\rho\nu}\partial_\sigma\xi^\nu + g_{\mu\sigma}\partial_\rho\xi^\mu 0 \tag{3.6}$$

Expanding this gives a set of simple first-order PDEs,

$$\partial_x \xi^x = 0; \partial_x \xi^y + \partial_y \xi^x = 0 \tag{3.7}$$

$$\partial_y \xi^y = 0; \partial_y \xi^z + \partial_z \xi^y = 0 \tag{3.8}$$

$$\partial_z \xi^z = 0; \partial_z \xi^x + \partial_x \xi^z = 0 \tag{3.9}$$

Using these, we can also conclude the following

$$\partial_y^2 \xi^x = \partial_z^2 \xi^x = 0 \tag{3.10}$$
$$\partial_x^2 \xi^y = \partial_z^2 \xi^y = 0 \tag{3.11}$$
$$\partial_x^2 \xi^z = \partial_y^2 \xi^z = 0 \tag{3.12}$$
$$\tag{3.13}$$

So up to the multiplication of possible constants, the possible KVFs are

$$\xi^x = 0 \text{ or } 1 \text{ or } \pm y \text{ or } \pm z \tag{3.14}$$
$$\xi^y = 0 \text{ or } 1 \text{ or } \pm x \text{ or } \pm z \tag{3.15}$$
$$\xi^z = 0 \text{ or } 1 \text{ or } \pm x \text{ or } \pm y \tag{3.16}$$
$$\tag{3.17}$$

That is, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(y, -x, 0)$, $(z, 0, -x)$, $(0, z, -y)$ are the 6 Killing Vector Fields for the Euclidean metric. These correspond to **all** the generators of translations and rotations in the Euclidean space!

Recall that these vector fields give us the directions in which we can traverse and still maintain the form of the metric. We see that in E^2 we could translate or rotate the points locally and still remain with the same metric. That is, we could *go anywhere on this space, and the metric remains the same*. We will have more to comment on this soon.

3.1.2 Sphere

We have the metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ on the sphere. The killing condition gives,

$$\partial_\theta \xi^\theta = 0 \tag{3.18}$$

$$2\sin^2\theta \partial_\phi \xi^\phi + \xi^\theta \sin 2\theta = 0 \tag{3.19}$$

$$\partial_\phi \xi^\theta + \sin^2\theta \partial_\theta \xi^\phi = 0 \tag{3.20}$$

A general solution to this system of PDE can be evaluated to be,

$$\xi = \big(A\cos\phi + B\sin\phi, -(A\sin\phi - B\cos\phi)\cot\theta + C\big) \tag{3.21}$$

with components being of θ and ϕ in order.

Importantly, this general solution can be written as a linear combination of the following three vector fields,

$$\xi_{(1)} = (\sin\phi, \cot\theta\cos\phi) \tag{3.22}$$

$$\xi_{(2)} = (\cos\phi, -\cot\theta\sin\phi) \tag{3.23}$$

$$\xi_{(3)} = (0, 1) \tag{3.24}$$

These correspond to two directions of translation and rotation around one axis on a 2-sphere. We thus have 3 Killing Vector Fields, which are essentially *all* the generators of translations and rotations on S^2.

3.1.3 Maximal Symmetry

There are essentially three important things-

1. The maximum number of linearly independent Killing Vectors (solutions to 3.5) for a
   d-dimensional space is
   $$d\,(d+1)\,/2.$$
2. A space is called Maximally Symmetric if the number of KVFs it holds is maximum.
3. The curvature tensor is fixed by the constant scalar curvature R (or K, $K = \frac{R}{D(D-1)}$).
4. Maximally symmetric spaces are uniquely specified by a curvature constant and by the number of eigenvalues of the metric that are positive(or negative).

Maximum Number of KVFs:

Informally, this corresponds to the total number of translational and rotational degrees of freedom available in the space. If the space is d-dimensional, then # translations = d, # rotations = $\frac{d(d-1)}{2}$. If we assume KVFs are either translational or rotational, then the maximum number is just the sum of translational and rotational ones!

Recall,

$$\left[\nabla_\mu, \nabla_\nu\right] V_\rho = -R^\sigma_{\rho\mu\nu} V_\sigma \tag{3.25}$$

and using the cyclic identity (Bianchi Identity) of the Curvature tensor, we have

$$\left[\nabla_\mu, \nabla_\nu\right] V_\rho + \left[\nabla_\nu, \nabla_\rho\right] V_\mu + \left[\nabla_\rho, \nabla_\mu\right] V_\nu = 0 \tag{3.26}$$

Applying these on ξ gives,

$$\nabla_\rho \nabla_\mu \xi_\nu = R^\tau_{\rho\mu\nu} \xi_\tau \tag{3.27}$$

We thus have the following system of PDE,

$$\nabla_\rho \xi_\mu = \chi_{\rho\mu} \tag{3.28}$$

$$\nabla_\rho \chi_{\mu\nu} = R^\tau_{\rho\mu\nu}\xi_\tau \tag{3.29}$$

where we have from 3.5, $\chi_{\rho\mu} = -\chi_{\mu\rho}$

This system takes $\frac{d(d+1)}{2}$ initial conditions, thus giving the same maximum number of independent solutions! We thus can have only $\frac{d(d+1)}{2}$ number of KVFs for any given d-dimensional space.

Curvature Tensor in a Maximally Symmetric Space:

Consider generalization of 3.25 to 2-rank tensors,

$$T_{\mu\rho;\nu\omega} - T_{\mu\rho;\omega;\nu} = -R^\tau_{\mu\nu\omega}T_{\tau\rho} - R^\tau_{\rho\nu\omega}T_{\mu\tau} \tag{3.30}$$

Where we have used the following notation for taking covariant derivatives,

$$\nabla_{\lambda_1}\nabla_{\lambda_2}\dots\nabla_{\lambda_n}\chi_{\mu\nu} = \chi_{\mu\nu;\lambda_1;\dots\lambda_n} \tag{3.31}$$

Let $T_{\mu\rho} = \nabla\mu\xi_\rho \equiv \xi_{\rho;\mu}$ using equation 3.29 we have $\xi_{\rho;\nu;\omega;\mu} = R^\tau_{\omega\mu\rho;\nu}\xi_\tau + R^\tau_{\omega\mu\rho}\xi_{\tau;\nu}$ and thus,

$$\xi_{\rho;\nu;\omega;\mu} - \xi_{\rho;\omega;\nu;\mu} = -R^\tau_{\mu\nu\omega}\xi_{\rho;\tau} - R^\tau_{\rho\nu\omega}\xi_{\tau;\mu} \tag{3.32}$$

$$\left(R^\tau_{\omega\mu\rho;\nu} - R^\tau_{\nu\mu\rho;\omega}\right)\xi_\tau = \left(R^\tau_{\nu\mu\rho}\delta^\kappa_\omega - R^\tau_{\omega\mu\rho}\delta^\kappa_\nu + R^\tau_{\mu\nu\omega}\delta^\kappa_\rho - R^\tau_{\rho\nu\omega}\delta^\kappa_\mu\right)\xi_{\tau;\kappa} \tag{3.33}$$

For translations, R.H.S of equation 3.33 vanishes, and we have (solving for linearly independent KVFs)

$$R^\tau_{\omega\mu\rho;\nu} = R^\tau_{\nu\mu\rho;\omega} \tag{3.34}$$

For rotation, L.H.S vanishes, implying that (solving for linearly independent KVFs again) the coefficient is a symmetric tensor. Then contracting κ with ρ and lowering τ gives,

$$R^{\tau}_{\nu\mu\rho}\delta^{\kappa}_{\omega} - R^{\tau}_{\omega\mu\rho}\delta^{\kappa}_{\nu} + R^{\tau}_{\mu\nu\omega}\delta^{\kappa}_{\rho} - R^{\tau}_{\rho\nu\omega}\delta^{\kappa}_{\mu} = R^{\kappa}_{\nu\mu\rho}\delta^{\tau}_{\omega} - R^{\kappa}_{\omega\mu\rho}\delta^{\tau}_{\nu} + R^{\kappa}_{\mu\nu\omega}\delta^{\tau}_{\rho} - R^{\kappa}_{\rho\nu\omega}\delta^{\tau}_{\mu} \tag{3.35}$$

$$R_{\tau\mu\nu\omega}(D-1) - \left(R_{\tau\nu\omega\mu} - R_{\tau\omega\mu\nu} - R_{\tau\mu\nu\omega}\right) = R_{\omega\mu}g_{\tau\nu} - R_{\nu\mu}g_{\tau\omega} \tag{3.36}$$

$$R_{\tau\mu\nu\omega} = \frac{R}{D(D-1)}\left(g_{\omega\mu}g_{\tau\nu} - g_{\nu\mu}g_{\tau\omega}\right). \tag{3.37}$$

Where we have used the cyclic property of curvature tensor and $R^{c}{}_{bca} = R^{c}{}_{acb} = -R^{c}{}_{abc} = R_{ab}$, $R^{c}{}_{cab} = 0$.

Now, use the (contracted) Bianchi identity $\nabla_{\mu}G^{\mu\nu} = 0$,

$$\nabla_{\mu}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) = \left(\frac{1}{D} - \frac{1}{2}\right)\nabla_{\mu}\left(Rg^{\mu}\nu\right) = 0 \tag{3.38}$$

$$\implies \left(\frac{1}{D} - \frac{1}{2}\right)\partial_{\mu}R = 0 \tag{3.39}$$

Since, $\nabla_{\alpha}g^{\beta\gamma} = 0$.

Thus for $D > 2$, we have a constant scalar curvature R. And for $D = 2$, use equation 3.34:

$$\left(g_{\omega\mu}\delta^{\tau}_{\rho} - g_{\omega\rho}\delta^{\tau}_{\mu}\right)\partial_{v}R = \left(g_{\nu\mu}\delta^{\tau}_{\rho} - g_{\nu\rho}\delta^{\tau}_{\mu}\right)\partial_{\omega}R \tag{3.40}$$

Contract τ with ρ and μ, ν with $g^{\mu\nu}$,

$$\left(2g_{\omega\mu} - g_{\omega\mu}\right)\partial_{v}R = \left(2g_{\nu}\mu - g_{\nu\mu}\right)\partial_{\omega}R \tag{3.41}$$

$$\delta^{\nu}_{\omega}\partial_{v}R = g_{\mu\nu}g^{\mu\nu}\partial_{\omega}R \tag{3.42}$$

$$\implies \partial_{\omega}R = 0 \tag{3.43}$$

We thus have a constant curvature for a maximal symmetric space, and the curvature tensor is fixed by it as,

$$R_{\tau\mu\nu\omega} = \frac{R}{D\,(D-1)} \left(g_{\omega\mu} g_{\tau\nu} - g_{\nu\mu} g_{\tau\omega} \right) \tag{3.44}$$

3.2 AdS as a maximally symmetric solution to Einstein Equation

We start with the Einstein-Hilbert Action,

$$S = -s \frac{1}{16\pi G_D} \int d^D \sqrt{|\, g \,|}\,(R - \Lambda) \tag{3.45}$$

giving the following vacuum Einstein equation with Cosmological Constant,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} \Lambda g_{\mu\nu} = 0. \tag{3.46}$$

Contracting with $g^{\mu\nu}$ we see that,

$$R = \frac{D}{D-2} \Lambda,\, R_{\mu\nu} = \frac{\Lambda}{D-2} g_{\mu\nu} \tag{3.47}$$

Such space with $R_{\mu\nu} \propto g_{\mu\nu}$ are called Einstein spaces. The Einstein equation's only condition on vacuum spaces is given by equations 3.47. That is a constant scalar curvature and $R_{\mu\nu \propto g_{\mu\nu}}$. We look for maximal symmetric solutions. As we will see, these spaces satisfy the above conditions without any further input with the Riemann tensor given by:

$$R_{\mu\nu\rho\sigma} = K \left(g_{\nu\sigma} g_{\mu\rho} - g_{\nu\rho} - g_{\mu\sigma} \right). \tag{3.48}$$

Anti-de Sitter space is the maximally symmetric solution to the Einstein Equations with a constant negative curvature and a negative cosmological constant. Similarly, we have de Sitter space with constant positive curvature and a positive cosmological constant.

3.2.1 AdS as an Embedding

We define the d-dimensional anti de Sitter space AdS_d with length scale L as the submanfiold of a $(d+1)$-dimensional Minkowski-type manifold $M^{d-1,2}$ as follows:

1. Let $\left(X^0, X^1, \ldots, X^d\right)$ be the coordinates on $M^{d-1,2}$.
2. So $M^{d-1,2}$ is a spacetime with $ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \ldots + \left(dX^{d-1}\right)^2 - \left(dX^d\right)^2$.
3. We define AdS_d as the set of all points $\left(X^0, X^1, \ldots, X^d\right)$ satisfying

$$-\left(X^0\right)^2 + (X^1)^2 + \ldots + (X^{d-1})^2 - (X^d)^2 = -L^2 \tag{3.49}$$

or,

$$(X^0)^2 - \sum_{i=1}^{d-1}(X^i)^2 + (X^d)^2 = L^2. \tag{3.50}$$

This embedding is carried out similarly to the sphere and de Sitter spaces. We see that, the isometry group of AdS^d is $SO(d-1,2)$ (with $(d+1)(d+2)/2$ number of generators) is thus a $(d+1)$ - dimensional maximally symmetric space!

Embedding Space	Metric	Submanifold	Definition of Submanifold	Isometry Group of the Submanifold
E^{d+1}	$ds^2 = \left(dX^0\right)^2 + \left(dX^1\right)^2 + \ldots + \left(dX^{d-1}\right)^2 + \left(dX^d\right)^2$	S^d	$\Sigma_{i=0}^{d}(X^i)^2 = L^2$	$SO(d+1)$
$M^{d,1}$	$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \ldots + \left(dX^{d-1}\right)^2 + \left(dX^d\right)^2$	dS^d	$-(X^0)^2 + \Sigma_{i=1}^{d}(X^i)^2 = L^2$	$SO(d,1)$
$M^{d-1,2}$	$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \ldots + \left(dX^{d-1}\right)^2 - \left(dX^d\right)^2$	AdS^d	$(X^0)^2 - \Sigma_{i=1}^{d-1}(X^i)^2 + (X^d)^2 = L^2$	$SO(d-1,2)$

Table 3.1: Sphere, de Sitter, and Anti-de Sitter Spaces.

3.2.2 Riemann Curvature Tensor

a) A Nifty Way: Locally Flat Metric:

Let $X^d = W$ and $X.X = \eta_{\mu\nu}X^\mu X^\nu$; $\mu, \nu = 0, 1, \ldots, d-1$ with $sign\left(\eta_{\mu\nu}\right) = (-1, 1, \ldots, 1)$. We can eliminate the W coordinate to get the coordinates on AdS^d.

The defnition of AdS^d now reads $X.X - W^2 = -L^2$.

So,

$$W^2 = L^2 + X.X \tag{3.51}$$

$$dW^2 = \frac{X.dX}{L^2 + X.X} \tag{3.52}$$

Thus,

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu - dW^2 \tag{3.53}$$

$$ds^2 = \left(\eta_{\mu\nu} - \left(\frac{\eta_{\mu\lambda}\eta_{\nu\rho}X^\lambda X^\rho}{L^2 + X.X} \right) \right) dX^\mu dX^\nu \tag{3.54}$$

Thus, for $X \rightarrow 0$, we have locally flat metric (at $X^\mu = 0$, where the Christophel symbols i.e the first order derivatives of g vanish) -

$$g_{\mu\nu} \approx \eta_{\mu\nu} - \frac{1}{L^2} \eta_{\mu\lambda}\eta_{\nu\rho} X^\lambda X^\rho \tag{3.55}$$

Thus it is now a lot easier to compute the curvature tensor:

For locally flat coordinates, $g_{\tau\mu}(x) = \eta_{\tau\mu} + B_{\tau\mu,\lambda\sigma} x^\lambda x^\sigma + \cdots$ we have -

$$R_{\tau\rho\mu\nu} = \left(B_{\tau\nu,\rho\mu} + B_{\rho\mu,\tau\nu} \right) - \left(B_{\rho\nu,\tau\mu} + B_{\tau\mu,\rho\nu} \right) \tag{3.56}$$

Using the form 3.55 we have

$$R_{\tau\rho\mu\nu} = -\frac{1}{L^2} \left(\eta_{\tau\mu}\eta_{\rho\nu} - \eta_{\tau\nu}\eta_{\rho\mu} \right) \tag{3.57}$$

Comparing this with the maximal symmetry condition 3.48 at $X^\mu = 0$ to get $K = -\frac{1}{L^2}$ and thus for d-dimensional anti-de Sitter space, we have at any arbitrary point,

$$R_{\tau\rho\mu\nu} = -\frac{1}{L^2}\left(g_{\tau\mu}g_{\rho\nu} - g_{\tau\nu}g_{\rho\mu}\right) \tag{3.58}$$

And

$$R = -\frac{d(d-1)}{L^2}, R_{\mu\nu} = -\frac{(d-1)}{L^2}g_{\mu\nu}, \Lambda = -\frac{(D-2)(D-1)}{L^2}. \tag{3.59}$$

b) Stereographic Coordinates: Conformally Flat Metric

We map $\left(X^0, X^1, \ldots, X^d\right)$ - coordinates on $M^{d-1,2}$ satisfying equation 3.50 to $\left(x^0, \ldots, x^d - 1\right)$ as follows:

$$X^\mu = \frac{x^\mu}{1 - \frac{x^2}{4\rho^2}}, \ \mu = 0, 1, \ldots, (d-1) \tag{3.60}$$

$$X^d = \rho\frac{1 + \frac{x^2}{4\rho^2}}{1 - \frac{x^2}{4\rho^2}} \tag{3.61}$$

where, $x^2 = -(x^0)^2 + (x^1)^2 + \ldots + (x^{d-1})^2$

Now, we write the metric on the embedding space in terms of these stereographic coordinates,

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{d-1}(dX^i)^2 - (dX^d)^2 \tag{3.62}$$

$$dX^d = d\rho\frac{1 + x^2}{1 - x^2} + 4\rho\frac{x_\mu dx^\mu}{\left(1 - x^2\right)^2} \tag{3.63}$$

$$dX^\mu = d\rho\frac{2x^\mu}{1 - x^2} + \frac{2\rho}{\left(1 - x^2\right)^2}\left[\left(1 - x^2\right)\delta^\mu_\nu + 2x^\mu x_\nu\right]dx^\nu \tag{3.64}$$

Then

$$ds^2 = d\rho^2 - \frac{4\rho^2}{(1 - x^2)^2}dx^2 \tag{3.65}$$

with $dx^2 = -(dx^0)^2 + (dx^1)^2 + \ldots + \left(dx^{d-1}\right)^2$

We thus see that AdS (of length scale L) spacetime is conformally flat!

$$g_\mu \nu = \left(\frac{1}{1 - \frac{x^2}{4L^2}}\right)^2 \eta_{\mu\nu}. \tag{3.66}$$

Now for a general conformally flat metric $g_{\mu\nu} = e^{\phi(x)}\eta_{\mu\nu}$, we have the following result:

$$R^\mu_{\nu\rho\sigma} = -\frac{4}{\left(1 - x^2\right)^2}\left(\eta_{\nu\sigma}\delta^\mu_\rho - \eta_{\nu\rho}\delta^\mu_\sigma\right) \tag{3.67}$$

$$= -\frac{1}{L^2}\left(-g_{\nu\rho}\delta^\mu_\sigma + g_{\nu\sigma}\delta^\mu_\rho\right) \tag{3.68}$$

3.3 Coordinates on AdS Spacetime and the Boundary

3.3.1 Angular Coordinates, Hyperbolic Space

The definition of AdS^3 is $(T^2+W^2)-(X^2+Y^2) = L^2)$ if (T, X, Y, W) are the coordinates of the embedding space.

Set

$$T = R\cos t, W = R\sin t \tag{3.69}$$

$$X = r\cos\theta, Y = r\sin\theta \tag{3.70}$$

Then

$$ds^2 = -\left(dR^2 + R^2 dt\right) + \left(dr^2 + r^2 d\theta^2\right) \tag{3.71}$$

We have,

$$R^2 - r^2 = 1$$

$$RdR = rdr$$

$$dR^2 - dr^2 = -\frac{1}{1+r^2}dr^2$$

$$ds^2 = -\left(1 + r^2\right) dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2 \tag{3.72}$$

This is independent of the time coordinate t viz. we have defined coordinates (t, r, θ) on AdS^3 where the metric is static.

$$\begin{aligned} T &= \sqrt{1 + r^2} \cos t \\ W &= \sqrt{1 + r^2} \sin t \\ X &= r \cos \theta \\ Y &= r \sin \theta \end{aligned} \tag{3.73}$$

Similarly, we have the following metric on AdS^d -

$$ds^2 = -\left(1 + r^2\right) dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-2}^2. \tag{3.74}$$

Now, letting $r = \sinh \rho$ -

$$\begin{aligned} ds^2 &= -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-2}^2 && (3.75) \\ &= -\cosh^2 \rho dt^2 + dH_{d-1}^2 && (3.76) \end{aligned}$$

This indicates that AdS^d has a **hyperbolic** spatial part!

$$\begin{aligned} T &= \cosh \rho \cos t \\ W &= \cosh \rho \sin t \\ X &= \sinh \rho \cos \theta \\ Y &= \sinh \rho \sin \theta \end{aligned} \tag{3.77}$$

3.3.2 Poincare Coordinates

Poincaré Half-Plane.

Consider the Upper Half plane of $\mathbb{R}^2$ equipped with following metric -

$$ds^2 = \frac{\left(dx^2 + dy^2\right)}{y^2} \tag{3.78}$$

There are two interesting features -

1. Consider 1d object along x axis situated at y - with length $l = \frac{\delta x}{y}$. Now as we decrease y, i.e. move towards $y = 0$, we see that the object shrinks (in x) [with constant l].

2. The edge/boundary y = 0 is infinitely far away from any (x, y)!

$$\int ds = \int_{0^+}^{y} \frac{dy}{y} = \log\left(\frac{y}{0^+}\right) \to \infty \tag{3.79}$$

This metric is again conformally flat and has hyperbolic geometry.

The concept of Poincaré Half Plane is useful in AdS, too, due to the following choice of coordinates:

Rewrite the definition of AdS^3: $(T^2 - X^2) + \left(W^2 - Y^2\right) = L^2$

Now write

$$\begin{aligned}
T^2 - X^2 &= L^2\frac{t^2 - x^2}{w^2} \\
W^2 - Y^2 &= L^2(1 + \frac{x^2 - t^2)}{w^2}) \\
T = L\frac{t}{w}, X &= L\frac{x}{w} \\
Y &= \frac{L}{2w}\left(x^2 - t^2 + w^2 - 1\right) \\
W &= \frac{L}{2w}\left(x^2 - t^2 + w^2 + 1\right)
\end{aligned} \tag{3.80}$$

This gives us the metric -

$$ds^2 = \frac{L^2}{w^2}\left(-dt^2 + dx^2 + dw^2\right) \tag{3.81}$$

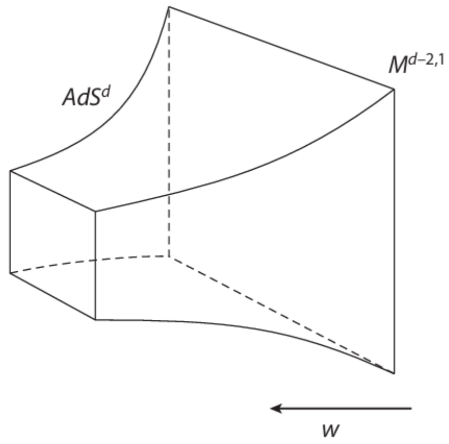

Figure 3.1: A constant w slice of AdS_d.

AdS^3 Minkowski version of Poincaré Half Plane equation 3.78! We thus have a spatial boundary at $w = 0$!

We can easily extend this to d-dimensions…

Light Cone Coordinates:

$$W^+ = W + Y = \frac{}{w}\left(x^2 - t^2\right) + w \tag{3.82}$$

$$W^- = W - Y = \frac{1}{w} \tag{3.83}$$

which satisfies $T^2 - X^2 + W^+W^- = L^2$

Consider the Poincaré coordinates on AdS^5:

$$ds^2 = \frac{L^2}{w^2}\left(-dt^2 + dx^2 + dy^2 + dz^2 + dw^2\right) \equiv \frac{L^2}{z^2}\left(-dt^2 + d\bar{x}^2 + dz^2\right) \tag{3.84}$$

Where in writing the second notation, we have let z denote the coordinate perpendicular to the boundary ($z = 0$). So slices of constant w are just the familiar four-dimensional Minkowski spacetimes $M^{3,1}$!

- Set $w = L^2/r$ to get

$$ds^2 = \left(-r^2dt^2 + \frac{1}{r^2}dr^2\right) + r^2d\bar{x}^2. \tag{3.85}$$

Here we have the boundary at $r \to \infty$

- Set $w = Le^{u/L}$ in equation 3.84 to get

$$ds^2 = e^{-\frac{2u}{L}} \left(-dt^2 + dx^2 + dy^2 + dz^2\right) + du^2 \tag{3.86}$$

with $u \in (-\infty, \infty)$.

We thus have Poincaré coordinates on AdS^d as:

$$X^\mu = e^{-\frac{u}{L}} x^\mu \tag{3.87}$$

$$X^+ = X^d + X^{d-1} = \frac{e^{-\frac{u}{L}}}{L} \eta_{\rho\sigma} x^\rho x^\sigma + Le^u \tag{3.88}$$

$$X^- = X^d - X^{d-1} = Le^{-\frac{u}{L}} \tag{3.89}$$

$$ds^2 = \eta_{\mu\nu} X^\mu X^\nu - dX^+ dX^- \tag{3.90}$$

giving the following metric -

$$ds^2 = e^{-\frac{2u}{L}} \eta_{\mu\nu} dx^\mu dx^\nu + du^2. \tag{3.91}$$

The splitting X^μ and $\left(X^d, X^{d-1}\right)$ reflects the splitting of the isometry group $SO\,(d-1,2)$ into its two subgroups - $SO\,(d-2,1)$ and $SO\,(1,1)$.

3.3.3 The boundary of AdS

Write $r = tan\psi$ in equation 3.74:

$$ds^2 = \frac{1}{\cos^2\psi} \left(-dt^2 + d\psi^2 + \sin^2\psi\, d\Omega^2_{d-2}\right) \tag{3.92}$$

Note that here $\psi \in \left[0, \frac{\pi}{2}\right)$ only (i.e. not upto π) and we thus have the spatial part not $d\Omega^2_{d-1}$ globally but only locally, i.e. the northern hemisphere. Thus spatial part is just B^{d-1}

topologically and is thus bounded by S^{d-2}! This is just E^{d-2} with infinity identified as a single point. Adding back the time coordinate, we see that AdS^d is bounded by $M^{d-2,1}$. In 3 dimensions, this is like a 'tin can' - a spatial part bounded by a circle and then time coordinate extending it to infinity.

Negatively curved spaces (or Hyperbolic spaces) do not have boundaries in general since they stretch to infinity. However, there is a notion of the conformal boundary for such spaces. We can illustrate this via AdS_3. The global metric reads,

$$ds^2 = \frac{L^2}{\cos^2 \rho}(-dt^2 + d\rho^2 + \sin^2 \rho d\theta^2) \tag{3.93}$$

Upon a conformal rescaling of this metric with $\cos^2 \rho / L^2$, we see that the spatial part corresponds to a half sphere having a boundary S^1 corresponding to $\rho \to \pi/2$. We conclude that asymptotically, the boundary(via conformal scaling) of AdS_3 is just $\mathbb{R} \times S^1$ - the cylinder.

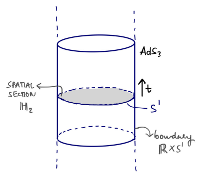

Figure 3.2: Geometry of AdS^3. The spatial part of the AdS bulk is hyperbolic in nature and is conformally a (half) sphere.

Given that we have a boundary based on conformal scaling, to connect to a theory in AdS bulk from the boundary, we need a *conformally invariant* theory on the boundary, which amounts to studying the Conformal Field Theories. In particular 2D CFTs on a cylinder play a significant role.

3.3.4 Hyperbolic Coordinates

On AdS^3:

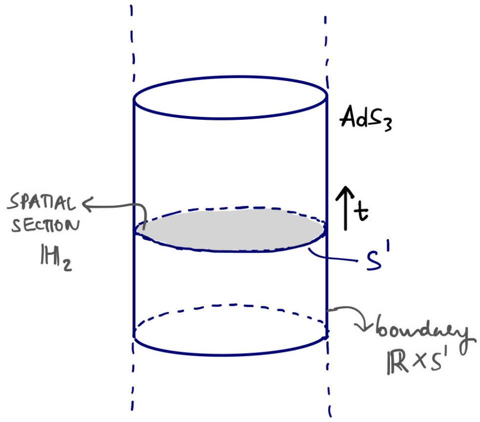

Figure 3.3: The geometry of AdS_3 with hyperbolic spatial section and $\mathbb{R} \times S^1$ conformal boundary. The interior of this cylinder is conformally related to the bulk of AdS_3.

$$(T^2 - X^2) + \left(W^2 - Y^2\right) = 1$$

Set,

$$\begin{aligned} T &= R\cosh t \\ X &= R\sinh t \\ W &= r\cosh\psi \\ Y &= r\sinh\psi \end{aligned} \tag{3.94}$$

and the metric becomes,

$$ds^2 = -\left(r^2 - 1\right)dt^2 + \frac{r^2}{r^2 - 1} + r^2 d\psi^2 \tag{3.95}$$

For AdS^d, this becomes -

$$ds^2 = -\left(r^2 - 1\right)dt^2 + \frac{dr^2}{r^2 - 1} + r^2 dH_{d-2}^2. \tag{3.96}$$

Note:

$$dH_d^2 = d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2 \tag{3.97}$$

$$d\Omega_d^2 = d\psi^2 + \sin^2 \psi d\Omega_{d-1}^2 \tag{3.98}$$

- For $S^3 \subset E^4$,

$$ds^2 = \frac{dr^2}{1-r^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right) \tag{3.99}$$

- For $S^2 \subset E^3$,

$$ds^2 = \frac{dr^2}{1-r^2} + r^2 d\varphi^2 \tag{3.100}$$

3.3.5 Euclidean Anti-de Sitter Space

Use the stereographic coordinates. Proceed by,

$$X^0 = ix^T \tag{3.101}$$

$$x^0 = ix^T \tag{3.102}$$

Then,

$$X^M = \frac{1}{1-\frac{x^2}{4L^2}} x^\mu, \; M = T, 1, 2, \ldots, d-1 \tag{3.103}$$

$$X^d = \frac{L\left(1+\frac{x^2}{4L^2}\right)}{1-\frac{x^2}{4L^2}} \tag{3.104}$$

Here $x = (x^T)^2 + (x^1)^2 + \ldots + (x^{d-1})^2$.

So that the Euclidean AdS condition is satisfied:

$$\left(X^T\right)^2 + \sum_{i=1}^{d-1} \left(X^i\right)^2 - (X^d)^2 = -L^2 \tag{3.105}$$

Thus giving,

$$ds^2 = \left(\frac{1}{1 - \frac{x^2}{4L^2}}\right)\left((dx^T)^2 + (dx^1)^2 + \ldots + \left(dx^{d-1}\right)^2\right). \tag{3.106}$$

- AdS_E^d is conformally related to E^d.
- AdS_E^d is toplogically the Euclidean ball $x^2 \leq 4L^2$, having a boundary S^{d-1} (which is basically E^{d-1} by identifying infinity with a single point).

3.3.6 Maldacena's:

In equation 3.50, set $u = X^0 + iX^d,\ v = X^0 - iX^d$ so that,

$$X^2 = X^+X^- - \bar{X}^2 = L^2 \tag{3.107}$$

Now, define the following

$$\xi^\alpha \equiv \frac{X^\alpha}{u};\ \alpha = 1, \ldots, d-1 \tag{3.108}$$

$$\bar{\xi}^2 \equiv \sum_{\alpha=1}^{d-1} (\xi^\alpha)^2 \tag{3.109}$$

Then using equation 3.107 eliminate v:

$$v = \bar{\xi}^2 u + \frac{L^2}{u} \tag{3.110}$$

Introduce the cooridnate (u, ξ^α) on AdS^d:

$$ds^2 = -L^2\frac{du^2}{u^2} + u^2 d\bar{\xi}^2 \tag{3.111}$$

3.3.7 Witten's

In equation 3.111 set $L = 1$ for simplicity and introduce the cooridnates

$$\left(\xi^0, \bar{\xi}\right) \equiv \left(u^{-1}, \bar{\xi}\right) \tag{3.112}$$

Then,

$$ds^2 = \frac{1}{(\xi^0)^2}\left(\left(d\xi^0\right)^2 + d\bar{\xi}^2\right) \tag{3.113}$$

This looks similar to that derived in Poincaré coordinates, but note that ξ^0 is a light cone coordinate, has both time and space mixed...

3.4 Penrose Diagrams

Penrsoe Diagrams offer a neat way to represent any spacetime into a finite region of space holding the some information of its causal structures as well. One can follow the following recipe to construct the Penrose Diagram which is also usually called Causal Diagram or a Conformal Diagram for reasons that will be clear from below:

1. Starting from any form of the metric of the given space-time, identify two non-compact coordinates.

2. Re-write the metric by using the null-coordinates over the previous two non-compact coordinates. The resultant will still be non-compact.

3. Compactify the two null-coorduinates (say using $\tan^{-1}$).

4. Re-define and go back to the temporal and space coordinates, which are now compact. Express the metric in these coordinates.

5. Remove the conformal factor and (symmetric) contribution from any other coordinates in the metric.

6. We end up with an unphysical Weyl transformed metric, whose two temporal and spatial coordinates are compactified, and the null geodesics (thus the causal structure) of the original spacetime are preserved.

Consider for example, the Minkowski Spacetime,

$$ds^2 = -dt^2 + dr^2 + r^2 d\omega^2 \tag{3.114}$$

with $t \in (-\infty, \infty)$ and $r \in [0, \infty)$.

1. We re-write this in the null coordinates as $u = t + r, v = t - r$ (also called the light-cone coordinates) with $u, v \in (-\infty, \infty)$-

$$ds^2 = -dudv + \cdots \tag{3.115}$$

2. To compactify u, v we define,

$$\begin{aligned} p &= \tan^{-1}(u) \\ q &= \tan^{-1}(v)\,. \end{aligned} \tag{3.116}$$

So, $p, q \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. And the metric now reads,

$$ds^2 = -\frac{1}{\cos^p \cos^q} dpdq + \cdots \tag{3.117}$$

3. We bring back the temporal coordinate via

$$\begin{aligned} T &= p + q \\ X &= p - q. \end{aligned} \tag{3.118}$$

These coordinates are compactified,

$$
\begin{aligned}
-\pi \leq T + X \leq \pi \\
-\pi \leq T - X \leq \pi .
\end{aligned}
\tag{3.119}
$$

The metric finally reads (after a weyl transformation to remove the conformal factors and supressing the contribution from other coordinates),

$$
ds^2 = -dT^2 + dX^2 \tag{3.120}
$$

Figure 3.4 shows the region of T, W with the null geodesics given by the $\frac{dT}{dW} = \pm 1$ lines.

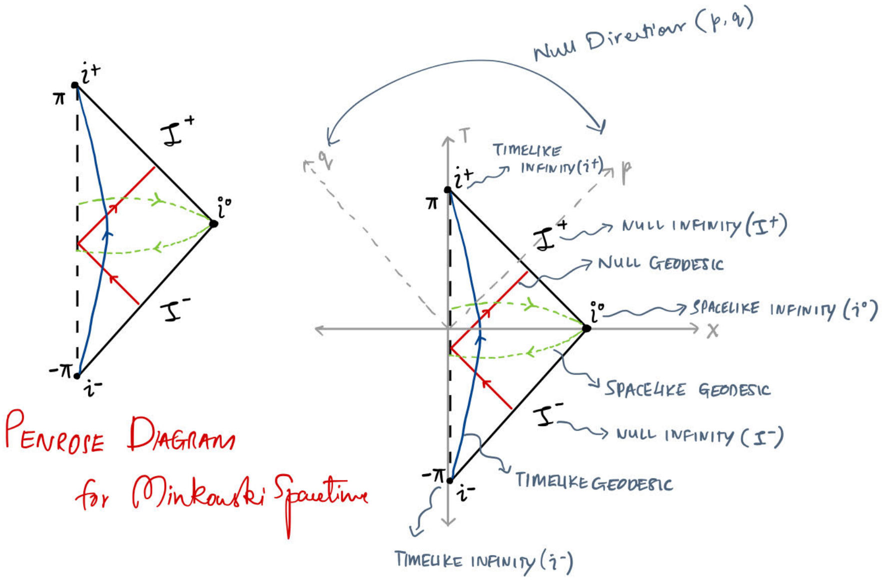

Figure 3.4: Note the suppressed S^2 at every point. This is the infamous half-diamond of Minkowski Space.One could add the S^1 part which makes the point i^0 into a circle. The boundary (null infinity) is the *conformal boundary* of the Minkowski Spacetime.

3.5 Geodesics and Penrose Diagram for AdS

Before drawing the conformal diagram, it's good to look at the geodesics on the AdS spacce-time. One can do so by extremizing the action of a free particle in a $d + 1$ dimensional space

($sign\,(-,+,\ldots,+,-)$) with coordinates X^{μ} along with a constraint (Lagrange Multiplier) $X^{\mu}X_{\mu} = L^2$, which restricts the particle to AdS.

$$S\left[X_{\mu},\lambda\right] = \int d\tau \left[\dot{X}^{\mu}\dot{X}_{\mu} + \lambda\left(L^2 - X^{\mu}X_{\mu}\right)\right] \tag{3.121}$$

This gives the following Equations of motion:

$$\ddot{X}^{\mu} = -\lambda X^{\mu} \tag{3.122}$$

$$X^{\mu}X_{\mu} = L^2 \tag{3.123}$$

Note that, upon scaling the world-line parameter $\tau \to \tilde{\tau} = \gamma\tau$:

$$S[X,\lambda] = \gamma \cdot S[X,\frac{\lambda}{\gamma^2}] \tag{3.124}$$

Thus, λ can be modified by any positive real γ^2. We thus have $\lambda = \pm\frac{1}{\gamma^2}, 0$ as the three kind of choices. Further, differentiating equation 3.123 gives, $\ddot{X}^m uX_{\mu} = -\dot{X}^{\nu}\dot{X}_{\nu}$. Thus $\lambda = +\frac{1}{\gamma^2}$ gives the timelike geodesics, $\lambda = -\frac{1}{\gamma^2}$ gives spacelike geodesics and $\lambda = 0$corresponds to the null geodesics in AdS.

1. **Timelike Geodesics:** $\ddot{X}^{\mu} = -\frac{1}{\gamma^2}X^{\mu}$, solutions of which are,

$$X^{\mu} = v^{\mu}\cos\frac{t}{\gamma} + \tilde{v}^{\mu}\sin\frac{t}{\gamma} \tag{3.125}$$

along with the constraint,

$$L^2 = v \cdot v\cos^2\frac{t}{\gamma} + \tilde{v}\cdot\tilde{v}\sin^2\frac{t}{\gamma} + v\cdot\tilde{v}\sin 2\frac{t}{\gamma} \tag{3.126}$$

- A trivial solution for the above is,

$$X_0 = L\cos\frac{t}{\gamma}, X_d = L\sin\frac{t}{\gamma}, X_i = 0. \tag{3.127}$$

In terms of the global coordinates, this is the trajectory for a rest particle at $\rho = 0$.

- One of the peculiar solutions is given by,

$$
\begin{aligned}
X_0 &= \frac{L\cos\frac{t}{\gamma}}{\cos\rho_*} \\
X_d &= L\sin\frac{t}{\gamma} \\
X_1 &= L\tan\rho_*\cos\frac{t}{\gamma} \\
X_i &= 0.
\end{aligned}
\tag{3.128}
$$

  These coordinates (not exactly in the global coordinates form) correspond to the oscillations around $\rho = \pm\rho_*$ in the 1-direction.

- A more general trajectory is given by,

$$
\begin{aligned}
X_0 &= \frac{L\cos\frac{t}{\gamma}}{\cos\rho_*} \\
X_d &= \frac{L\sin\frac{t}{\gamma}}{\cos\rho_*} \\
X_1 &= L\cos\frac{t}{\gamma}\tan\rho_* \\
X_2 &= L\sin\frac{t}{\gamma}\tan\rho_* \\
X_i &= 0.
\end{aligned}
\tag{3.129}
$$

  This trajectory describes the particle in global coordinates, circuling at $\rho = \rho_*$, with $\theta(t) = t$ in the 1-2 plane.

2. **Null Geodesics:**$\ddot{X}^\mu = 0$. The solutions are just linear. The geodesics of Minkowski space are indeed preserved.

3. **Spacelike Geodesics:** $\ddot{X}^\mu = \frac{1}{\gamma^2}X$. Solutions take the form of hyoerbolic functions.

Some facts to note:

1. Any timelife geodesic on AdS spacetime is equivalent to a circular orbit or a particle at rest via a suitable conformal transformation (AdS) Isometry.

2. All timelike geodesics run at same frequency $\frac{1}{\gamma}$, with period $2\pi\gamma$.

3. A null geodesic takes a finite time to reach the boundary (which is infinitely far away) - thus requiring a boundary condition. One usually assumes a reflective boundary condition.

Now we try to draw the penrose diagram for AdS keeping the above in mind. In the global coordinates the (Weyl transformed) metric reads,

$$ds^2 = dt^2 - d\rho^2 - \sin^2 \rho d\Omega^2 \tag{3.130}$$

with $t \in (-\infty, \infty)$ and $\rho \in [0, \frac{\pi}{2})$.

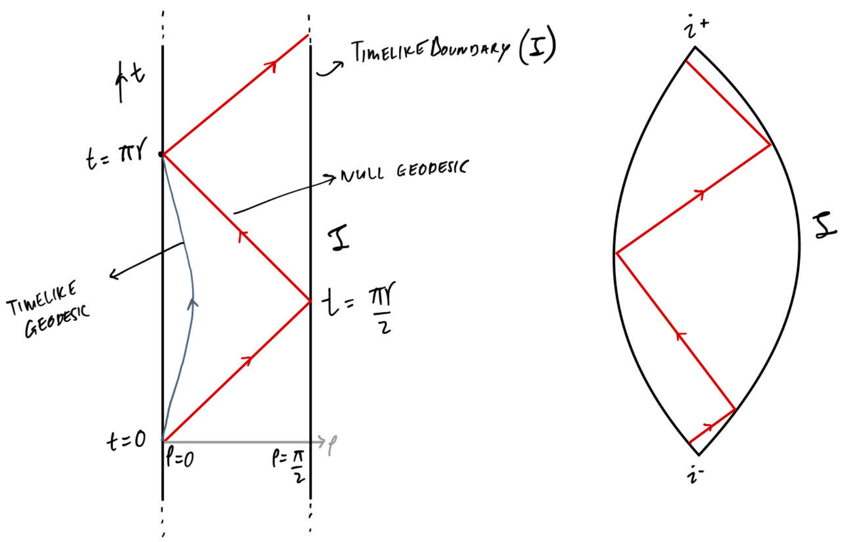

Figure 3.5: Penrose Diagram for AdS. On the right the time coordinate is compactified. The timelike boundary is $\mathbb{R} \times S^1$ for AdS^3.

3.6 Approaching the Boundary

We have seen two different metrics on the (conformal) boundary of AdS - a *cylindrical* one from the global coordinates and a *Minkowski* one from Poincaré Coordinates. The way to reconcile this is to note that there are, in fact, infinitely many boundary metrics one can approach from the bulk of AdS. Starting from the global coordinates, we can see it as follow. One can approach the boundary $\rho = \frac{\pi}{2}$ generally via,

$$\rho = \frac{\pi}{2} - \epsilon f(t, \Omega) \tag{3.131}$$

taking $\epsilon \to 0$ with $f(t, \Omega)$ being an arbitrary function. The metric is modified as,

$$\begin{aligned} ds^2 &= \frac{1}{\cos^2(\rho(\epsilon, f))}\left(-dt^2 + \sin^2(\rho(\epsilon, f))\, d\Omega_i^2\right) \\ &\approx \frac{1}{\epsilon^2 f(t, \Omega)}\left(-dt^2 + (1-\epsilon)\, d\Omega_i^2\right) \\ &\xrightarrow{\epsilon \to 0} \left[f(t, \Omega)\right]^{-2}\left(-dt^2 + d\Omega^2\right) \end{aligned} \tag{3.132}$$

Take $f = e^{-t}$ and then define $r = e^t$, get back the flat space coordinates (polar form). All these are indeed conformally related to the flat metric. For Euclidean AdS, $\rho = \frac{\pi}{2} - \epsilon e^{-t}$ gives the flat **Euclidean Boundary**. $\rho = \frac{\pi}{2} - \epsilon \cos(t)$ gives a **de Sitter type Boundary**. $\rho = \frac{\pi}{2} - \frac{z}{L^2}(\cos\theta - \cos t)$ gives the flat **Lorentzian Boundary**. Where, $z \to 0$! We cleverly used the Poincaré coordinates where z now fulfils the role of ϵ!

3.6.1 Projecive Null Cone

We can define coordinates on the boundary of AdS by using the embedding space. On AdS, we had for the Euclidean case,

$$x_0 = \frac{L}{\cos h\rho}\cos t, \quad W = \frac{L}{\cosh\rho}\sin t \tag{3.133}$$

$$X_i = L\tan\rho\Omega_i, \quad \rho \in (0, \pi/2) \tag{3.134}$$

At the boundary (approaching through 3.131), $X_A \to \infty$. We can instead define $P_A = \epsilon X_A$ on the boundary, which makes P_A finite ($\epsilon \to 0$). Since $X^A X_A = L^2$, we have $P_A P^A = 0$ and also $P_A \sim \lambda P_A$. These coordinates define the projective null cone - the boundary of Euclidean AdS in the embedding space. One can then explicitly find the geometry of the boundary - as a section of the null projective cone corresponding to different fs.

Chapter 4

Symmetries in Conformal Field Theory

In the last chapter, we saw that the boundary metric of AdS is defined only up to a conformal scaling. This is how conformal field theory enters the theory in AdS, as a potential QFT on the boundary. Conformal field theory is a quantum field theory that is symmetric under conformal transformations in addition to Lorentz or Poincaré transformations. The goal of the current chapter is to describe the conformal transformations (as a generalisation to isometries) and the algebra (group) formed by them in general dimensions. We also study the fundamental properties of QFTs with such symmetries. In particular, we study the behaviour of fields, constraints on the stress tensor and the correlation functions in general dimensions.

4.1 Conformal Transformations

Transformations $x'^{\mu} = x^{\mu} + \epsilon\xi^{\mu}(x)$ which satisfy

$$g'_{\rho\sigma}(x') = \Lambda(x) g_{\rho\sigma}(x) \tag{4.1}$$

This gives us the conformal killing condition,

$$\xi_{\sigma;\rho} + \xi_{\rho;\sigma} + \kappa g_{\rho\sigma} = 0 \tag{4.2}$$

where we let $\Lambda(x) \approx 1 + \epsilon\kappa(x) + O\left(\epsilon^2\right)$.

In flat spacetime, this gives,

$$\partial_\rho \xi_\sigma + \partial_\sigma \xi_\rho = \frac{2}{d}\eta_{\rho\sigma}\partial \cdot \xi \tag{4.3}$$

A little more calculation and we can completely determine ϵ_μ,

$$\xi_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho;\ c_{\mu\nu\rho} = c_{\mu\rho\nu}. \tag{4.4}$$

i.e. it can depend on x at most quadratically.

Subbing this form order by order, we get the following:

1. No restriction on the constant term a_μ.
2. $b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu};\ m_{\mu\nu} = -m_{\nu\mu}$. α is the trace of b. We thus have a pure trace term (corresponding to the scaling) and a Lorentz transformation term in $b_{\mu\nu}$.
3. $c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu;\ b_\mu = \frac{1}{\alpha} c^\sigma_{\sigma\mu}$

We then have the following:

Transformation type	Finite Transformation	Infinitesimal Generator (for a scalar field) ('G_a')
Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Dilatation/Scaling	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu\partial_\mu$
Rotation	$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
Special Conformal Transformation	$x'\mu = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2$ $\downarrow$ $x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot.x+b^2x^2}$	$K_\mu = -i(2x_m u x^\nu\partial_\nu - x^2\partial_\mu)$

Table 4.1: Conformal Transformations, their finite forms, and infinitesimal generators for (scalar) fields.

The infinitesimal and finite transformations are familiar in the cases of translation, rotation, and scaling. The calculation for SCT is given in A.4.

4.2 Conformal Algebra

The conformal algebra is then just extended from Poincaré algebra adding the commutation rules with D, K as follows:

$$\left[D, P_\mu\right] = iP_\mu \tag{4.5}$$

$$\left[D, K_\mu\right] = -iK_\mu \tag{4.6}$$

$$\left[D, L_{\mu\nu}\right] = 0 \tag{4.7}$$

$$\left[K_\mu, P_\nu\right] = 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right) \tag{4.8}$$

$$\left[K_\mu, K_\nu\right] = 0 \tag{4.9}$$

$$\left[K_\rho, L_{\mu\nu}\right] = i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right) \tag{4.10}$$

$$\left[L_{\mu\nu}, P_\rho\right] = i\left(\eta_{\rho\nu}P_\mu - \eta_{\rho\mu}P_\nu\right) \tag{4.11}$$

$$\left[L_{\mu\nu}, L_{\rho\sigma}\right] = i\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}\right) \tag{4.12}$$

Write the generators as follows:

$$J_{\mu\nu} = L_{\mu\nu} \tag{4.13}$$

$$J_{-1,0} = D \tag{4.14}$$

$$J_{-1,\mu} = \frac{1}{2}(\left(P_\mu - K_\mu\right) \tag{4.15}$$

$$J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right) \tag{4.16}$$

Here, $\mu = 1, 2, \ldots, d$ and $J_{ab} = -J_{ab}$ for $a, b = -1, 0, 1, \ldots, d$.

Now comes the fun part!

J_{ab} satisfies the following algebra,

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right). \tag{4.17}$$

Yes, this is just the $SO\,(d, 2)$ lie algebra if we consider the space to be $M^{d-1,1}$ which has $SO\,(d-1, 1)$ as its isometry group! And similarly, this forms an $SO\,(d+1, 1)$ algebra on Euclidean space with isometry group $SO(d)$! *Note the (+1,+1) while moving from the isometry group to the conformal group!)*

Poincaré + Dilations together form a sub-algebra of the full conformal group. In general, Poincaré and dilation invariance doesn't imply Conformal Invariance, but there are special cases where conformal invariance follows.

4.3 Conformal Invariance in Classical Field Theory

Let $S_{\mu\nu}, \tilde{\Delta}, \kappa_\mu$ be a representation of the following reduced algebra of the full conformal algebra:

$$
\begin{aligned}
\left[\tilde{\Delta}, S_{\mu\nu}\right] &= 0 \\
\left[\tilde{\Delta}, \kappa_\mu\right] &= -i\kappa_\mu \\
\left[\kappa_\mu, \kappa_\nu\right] &= 0 \\
\left[\kappa_\rho, S_{\mu\nu}\right] &= i\left(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu\right) \\
\left[S_{\mu\nu}, S_{\rho\sigma}\right] &= i\left(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\right)
\end{aligned} \tag{4.18}
$$

So that,

$$
\begin{aligned}
L_{\mu\nu}\phi(0) &= S_{\mu\nu}\phi(0) \\
D\phi(0) &= \tilde{\Delta}\phi(o) \\
K_\mu\phi(0) &= \kappa_\mu\phi(0)
\end{aligned} \tag{4.19}
$$

Using the BCH formula and the reduced algebra 4.18, we compute the action of the conformal generators on the fields at arbitrary positions. Note that translation is the only transformation that doesn't preserve $x^\mu = 0$.

$$\begin{aligned} L_{\mu\nu}(s) &= e^{ix^\rho P_\rho L_{\mu\nu}(0)} e^{-ix^\rho P_\rho} \\ D(x) &= e^{ix^\rho P_\rho} D(0)\, e^{-ix^\rho P_\rho} \\ K_\mu(x) &= e^{ix^\rho P_\rho} K_\mu(0) e^{-ix^\rho P_\rho} \end{aligned} \tag{4.20}$$

Then we have the action on fields as follows:

$$P_\mu \phi(x) = -i\partial_\mu \phi(x) \tag{4.21}$$

$$D\phi(x) = \left(-ix^\nu \partial_\nu + \tilde{\Delta}\right) \phi(x) \tag{4.22}$$

$$K_\mu \phi(x) = \left\{\kappa_\mu + 2x_\mu \tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\right\} \phi(x) \tag{4.23}$$

$$L_{\mu\nu}\phi(x) = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \phi(x) + S_{\mu\nu}\phi(x) \tag{4.24}$$

These are basically the G_a in $\phi'(x') = (1 - i\omega_a G_a)\,\phi(x) = \mathcal{F}\phi(x)$, computed using the reduced algebra and translation rather than going through the calculation, like in deriving A.30.

We can determine the forms of $\tilde{\Delta}, \kappa_\mu$ by letting the fields $\phi(x)$ belong to an irreducible representation of the Lorentz Group.

Schur's Lemma: Suppose a group G has an irreducible representation $R_(g)$ for $g \in G$. Then, if a matrix A commutes with all $R(g)$, then A is a multiple of the Identity matrix.

$$AR(g) = R(g)A \forall g \in G \implies A = \lambda I, \text{for some} \lambda.$$

.

Then by 4.18 we get

$$\tilde{\Delta} \propto I, \kappa_\mu = 0. \tag{4.25}$$

We can connect the scaling dimension to this via,

$$\tilde{\Delta} = -i\Delta. \tag{4.26}$$

We will work out the details in the next section.

We are now equipped to answer the question, how do fields behave under a general conformal transformation?

We will answer this here but discuss the details again in the next section.

The scalar fields transform as,

$$\phi(x) \longrightarrow \phi'(x') = \mid \frac{\partial x'}{\partial x} \mid^{-\frac{\Delta}{d}} \phi(x) \tag{4.27}$$

where, $\mid \frac{\partial x'}{\partial x} \mid = \Lambda(x)^{-\frac{d}{2}}$ - the Jacobian of conformal transformations. Recall $\Lambda(x)$ from equation 4.1.

The fields behaving like above under conformal transformations are called *Quasi Primary Fields*.

4.3.1 Quasi-Primary Fields: A derivation

One can derive the transformation of fields under general conformal transformations. We could ask the same question for the Lorentz group as well. How do the fields behave under them? We generally write $\phi'(x') = D\phi(x)$ where D is some representation of the Lorentz Group on the space of fields, generated by $S^{\rho\nu}$ satisfying the Lorentz Lie algebra. used this to derive the generators for fields that give us this finite form! After all, we were in the spell of 'Particles are an irreducible representation of the Poincaré group'. So are the fields of those particles. In our case of CFT, the representation of the Conformal Group is almost known because the fields we consider are irreducible representations of the Lorentz group and the rest of the details follow from Schur's lemma; we just need to use the generators to derive the full finite form.

Consider the full conformal transformation,

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} + \epsilon \left\{ a^{\mu} + x^{\mu} + \omega^{\mu}_{\nu} x^{\nu} + 2(b \cdot x) x^{\mu} - b^{\mu} x^2 \right\} \\
\implies \dot{x}^{\mu} &= a^{\mu} + x^{\mu} + \omega^{\mu}_{\nu} x^{\nu} + 2(b \cdot x) x^{\mu} - b^{\mu} x^2
\end{aligned} \tag{4.28}
$$

A general theory of symmetries in QFT (behaviour of fields) is discussed in the appendix A. For the behaviour of the field under the above general conformal transformation, we just need to include all the terms using 4.21- 4.24, according to equation A.21.

$$
\phi'(x) = \phi(x) + \epsilon \left\{ -i a^{\mu} P_{\mu} - iD - \frac{i}{2} \omega^{\rho\nu} L_{\rho\nu} - i b^{\mu} K_{\mu} \right\} \tag{4.29}
$$

So,

$$
\begin{aligned}
\dot{\phi}(x) = \Bigg[&- a^{\mu} \partial_{\mu} - x^{\mu} \partial_{\mu} - i\tilde{\Delta} + \frac{1}{2} \omega^{\rho\nu} \left(x_{\rho} \partial_{\nu} - x_{\nu} \partial_{\rho} \right) - \frac{i}{2} \omega^{\rho\nu} S_{\rho\nu} \\
&+ b^{\mu} \left(-i\kappa_{\mu} - 2i x_{\mu} \tilde{\Delta} + x^{\nu} S_{\mu\nu} - 2 x_{\mu} x^{\nu} \partial_{\nu} + x^2 \partial_{\mu} \right) \Bigg] \phi(x)
\end{aligned} \tag{4.30}
$$

Now, for a spinless $\phi(x)$ let

$$
\tilde{\Delta} = -i \Delta I, \ \kappa_{\mu} = 0, S_{\mu\nu} = 0.
$$

Then, the infinitesimal flow for the full conformal transformation is,

$$
\dot{\phi}(x) = \left[- a^{\mu} \partial_{\mu} - x^{\mu} \partial_{\mu} - \Delta + \frac{1}{2} \omega^{\rho\nu} \left(x_{\rho} \partial_{\nu} - x_{\nu} \partial_{\rho} \right) + b^{\mu} \left(-2 x_{\mu} \Delta - 2 x_{\mu} x^{\nu} \partial_{\nu} + x^2 \partial_{\mu} \right) \right] \phi(x) \tag{4.31}
$$

Note that we are interested to know the relation between $\phi(x')$ and $\phi(x)$, i.e $\mathcal{F}(\phi(x))$ (see A). A lot of the intricacy in the above equation comes from comparing fields at the same points. If we were to compare them at different points, which essentially translates to finding $\mathcal{F}$, then many terms might cancel out to give a less dauntingly looking flow.

Let's work it out for the known case of Lorentz transformations first. Assume we do not know $\mathcal{F}$, but that we do know the generator $L_{\rho\nu}$ given by equation 4.24.

In principle, for a scalar field, we should get $\mathcal{F}(\phi(x))$ to be just $\phi(x)$. Let's see. We will try to do this in the ϵ-variation form.

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \tag{4.32}$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \tag{4.33}$$

$$\phi'(x) = \phi(x) - i\omega_a G_a \phi(x) \tag{4.34}$$

The infinitesimal nature of ω can be captured by ϵ, allowing us to write the flow form as follows

$$x'^{\mu} = x^{\mu} + \epsilon\omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \tag{4.35}$$

$$\dot{x}^{\mu} = \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \tag{4.36}$$

$$\phi(x)(\epsilon) = \phi(x) - i\epsilon\omega_a G_a \phi(x) \tag{4.37}$$

$$\dot{\phi}(x) = -i\omega_a G_a \phi(x) \tag{4.38}$$

$$\phi\left(x(\epsilon), \epsilon\right) = \phi(x) + \epsilon\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \tag{4.39}$$

$$\frac{d}{d\epsilon}\phi\left(x(\epsilon), \epsilon\right) = \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \tag{4.40}$$

equation 4.40 is intersting. Use the chain rule and find that,

$$\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} = \dot{\phi}(x) + \dot{x}^{\mu}\partial_{\mu}\phi(x) \tag{4.41}$$

Thus in the ϵ-variation form, the finite transformation of the fields is,

$$\frac{d}{d\epsilon}\phi\left(x(\epsilon), \epsilon\right) = \dot{\phi}(x) + \dot{x}^{\mu}\partial_{\mu}\phi(x) \tag{4.42}$$

where to recall again,

$$\dot{x}^{\mu} = \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \tag{4.43}$$

$$\dot{\phi}(x) = -i\omega_a G_a \phi(x) \tag{4.44}$$

Note that all the terms in the RHS have their ϵ dependence, too; we dropped it for convenience.

Now to the Lorentz transformations. It should be simple - we expect the RHS to vanish.

$$\dot{x}^{\mu} = \frac{1}{2}\Omega^{\rho\sigma}\left(\delta^{\mu}_{\rho}x_{\sigma} - \delta^{\mu}_{\sigma}x_{\rho}\right) \tag{4.45}$$

$$\dot{\phi}(x) = -\frac{i}{2}\Omega^{\rho\sigma}\left(i\left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right)\right) \tag{4.46}$$

Clearly,

$$\dot{x}^{\mu}\partial_{\mu} \sim x_{\sigma}\partial_{\rho} - x_{\rho}\partial_{\sigma}$$

$$\dot{\phi}(x) \sim x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}$$

And they indeed cancel out! Our guess for a less daunting flow form is correct . . .. Basically, our generators include both the functional change and the argument (coordinate) change. So things simplify when we put everything together. [Actually, recall in the Lorentz case, the generators were determined so that when you put everything together, things are simpler, i.e. give out a representation of the Lorentz Group]

We can now tackle the full conformal group. Use equations 4.28 and 4.31 in 4.42,

$$\begin{aligned}\frac{d}{d\epsilon}\phi\left(x(\epsilon),\epsilon\right) = \Bigg[&-a^{\mu}\partial_{\mu} - x^{\mu}\partial_{\mu} - \Delta + \frac{1}{2}\Omega^{\rho\sigma}\left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right)\\ &+ b^{\mu}\left(-2x_{\mu}\Delta - 2x_{\mu}x^{\nu}\partial_{\nu} + x^2\partial_{\mu}\right)\Bigg]\phi(x)\\ &+\left[a^{\mu} + x^{\mu} + \frac{1}{2}\Omega^{\rho\sigma}\left(\delta^{\mu}_{\rho}x_{\sigma} - \delta^{\mu}_{\sigma}x_{\rho}\right) + 2(b\cdot x)x^{\mu} - b^{\mu}x^2\right]\partial_{\mu}\phi(x)\end{aligned} \tag{4.47}$$

Clearly, this is just

$$\begin{aligned}\frac{d}{d\epsilon}\phi\left(x(\epsilon),\epsilon\right) &= -\Delta\phi(x) - 2b\cdot x\Delta\phi(x)\\ &= -\Delta\left(1 + 2b\cdot x\right)\phi(x)\end{aligned} \tag{4.48}$$

Let's try to relate this to the conformal scaling factor of the metric $\Lambda(x)$ in equation 4.1, which technically dictates/defines the whole conformal transformation. From the analysis given in the section 4.1, we write

$$\begin{aligned}\Lambda'(x') &\approx 1 + \epsilon\kappa(x)\\ \kappa(x) &= -\frac{2}{d}\partial \cdot \xi\end{aligned} \tag{4.49}$$

Using the ξ for the full conformal transformation, we get

$$\begin{aligned}\Lambda(x) &= 1 + \epsilon\left(-\frac{2}{d}\left(d + 2(b.x)(1+d) - 2(b.x)\right)\right)\\ &= 1 - \epsilon\left[2\left(1 + 2b \cdot x\right)\right]\end{aligned} \tag{4.50}$$

Looks like it is indeed connected. So for general conformal transformation, we have (expanding $\Lambda(x) = 1 + \epsilon\kappa(x)$)

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x) = 1 - \epsilon\left[-2\left(1 + 2b \cdot x\right)\right] g_{\mu\nu} \tag{4.51}$$

so we have on the same lines of equations 4.39 and 4.40 the following evolution of the metric -

$$\frac{d}{d\epsilon}g_{\mu\nu}(x(\epsilon),\epsilon) = -2\left(1 + 2b \cdot x\right) g_{\mu\nu}(x(\epsilon),\epsilon) \tag{4.52}$$

Using this in equation 4.48 gives,

$$\begin{aligned}\frac{\dot{\phi}}{\phi} &= \frac{\Delta}{2}\frac{\dot{g_{\mu\nu}}}{g_{\mu\nu}}\\ \phi'(x') &= \left(\frac{g'_{\mu\nu(x')}}{g_{\mu\nu(x)}}\right)^{\Delta/2}\phi(x) = \Lambda(x)^{\Delta/2}\phi(x).\end{aligned} \tag{4.53}$$

We thus have the general conformal transformation law for spinless fields.

4.3.2 Energy Momentum Tensor

Now, under conformal transformations,

$$\delta S = \frac{1}{d} \int d^d x T^\mu_\mu \left(\partial \cdot \epsilon\right) \tag{4.54}$$

1. Tracelessness of $T^{\mu\nu} \implies$ Invariance of Action under Conformal Transformations! And the converse need not be true since ϵ is not arbitrary.

2. Under certain conditions, $T^{\mu\nu}$ of a scale invariant theory can be made traceless.

Consider $x'^\mu = (1 + \alpha)\, x^\mu$; $\mathcal{F}\left(\phi\right) = (1 - \alpha\Delta)\, \phi$ - the scale transformation.

Then we have,

$$j^\mu_D = T_c{}^\mu{}_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Delta . \phi, \tag{4.55}$$

here, T_c is the canonical Stress tensor - due to translations.

Define the *Virial* of the field ϕ to be:

$$V^\mu = \frac{\delta \mathcal{L}}{\delta(\partial^\rho \phi)} \left(\eta^{\mu\rho}\Delta + i S^{\mu\rho}\right) \phi \tag{4.56}$$

For a scale-invariant theory if it possible to write

$$V^\mu = \partial_\alpha \sigma^{\alpha\mu} \tag{4.57}$$

for some $\sigma^{\alpha\mu}$, then let:

$$\sigma^{\mu\nu}_+ = \frac{1}{2}\left(\sigma^{\mu\nu} + \sigma^{\nu\mu}\right) \tag{4.58}$$

so that defining,

$$X^{\lambda\rho\mu\nu} = \frac{2}{(d-2)} \left\{ \eta^{\lambda\rho}\sigma^{\mu\nu}_+ - \eta^{\lambda\mu}\sigma^{\rho\nu}_+ - \eta^{\lambda\mu}\sigma^{\nu\rho}_+ + \eta^{\mu\nu}\sigma^{\lambda\rho}_+ + \frac{1}{d-1}\left(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu}\right)\sigma_+{}^\alpha{}_\alpha \right\} \tag{4.59}$$

allows us to modify $T^{\mu\nu}$ as follows:

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} \tag{4.60}$$

The first two terms just correspond to the Belifante Tensor, and the third is symmetric in μ, ν and $\partial_\mu \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 0$. The trace of the third term is $\partial_\mu V^\mu$.

We then have,

$$T_\mu^\mu = \partial_\mu j_D^\mu \tag{4.61}$$

which vanishes on the shell! And we can thus write,

$$j_D^\mu = T_\nu^\mu x^\nu \tag{4.62}$$

ad the current corresponding to Dilation. But this analysis is clearly valid only for $d \neq 2$. For a $d = 2$ free field, the canonical/Belifante $T^{\mu\nu}$ has a vanishing trace! So, no modification is required! There is no general proof known in 2 dimensions regarding the traclessness for scale-invariant theories. But we will assume it to be true and continue. We will show that the expectation value of T_μ^μ vanishes in $d = 2$ if conformal invariance is present to justify this.

4.4 Conformal Symmetry and Correlation Functions

Now, we move to the behaviour of correlation functions under conformal transformations.

> The richness of conformal invariance in 2 Dimensions allows us to define theories based solely on the symmetry properties of the correlation functions, without reference (except in a few cases) to an action or a functional integral! ([7])

It speaks the strength of the conformal symmetry. Rather than counting the independent degrees of CFT, we focus on the number of local operators closed under conformal transformations.

4.4.1 2-point Correlators

We have for Quasi-Primary, spinless fields,

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{d}} \langle \phi_1(x_1')\phi_2(x_2')\rangle \tag{4.63}$$

- Scale Inv. $\implies$ a prefactor of $\lambda^{\Delta_1+\Delta_2}$.
- Rotation and translation Inv. further restricts the correlator to $f(\mid x_1 - x_2 \mid)$, such that $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$.

So thus far, we can say that,

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{\mid x_1 - x_2 \mid^{\Delta_1+\Delta_2}} \tag{4.64}$$

Now, let's analyze what SCT does -

Firstly, note that under SCT,

$$\mid x_1 - x_2 \mid \rightarrow \frac{\mid x_1 - x_2 \mid}{\gamma_1^{\frac{1}{2}}\gamma_2^{\frac{1}{2}}} \tag{4.65}$$

where,

$$\mid \frac{\partial x'}{\partial x} \mid_{SCT} = \frac{1}{\left(1 - 2b\cdot x + b^2x^2\right)^d} = \gamma^{-d} \tag{4.66}$$

So, $\gamma_{x=x_1} = \gamma_1$, $\gamma_{x=x_2} = \gamma_2$.

So for respecting equation 4.63 we must have

$$\frac{C_{12}}{\mid x_1 - x_2 \mid^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\mid x_1 - x_2 \mid^{\Delta_1+\Delta_2}} \frac{\left(\gamma_1\gamma_2\right)^{(\Delta_1+\Delta_2)/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}$$

To satisfy this identically - we must have $\Delta_1 = \Delta_2$.

Hence, the 2-point correlator for a conformally invariant theory is completely restricted up to a constant!

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{4.67}$$

4.4.2 3-point Correlator

The Scale, translation and rotation impose a similar restriction again,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c}$$

st. $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. Note that the above can be summed over a, b, c.

But SCT imposes the following -

$$\frac{C_{123}^{(abc)}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} = \frac{C_{123}^{(abc)}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{\frac{a}{2}}(\gamma_2\gamma_3)^{\frac{b}{2}}(\gamma_3\gamma_1)^{\frac{c}{2}}}{x_{12}^a x_{23}^b x_{31}^c}$$

Thus, the three-point correlator is then completely restricted by a constant again as follows:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{4.68}$$

Such a complete evaluation of the form of correlation functions stops at 3-point functions. For, there are cross-ratios, which are conformal invariants, possible in higher correlators.

4.4.3 4pt. Correlator

If we have four points, x_1, x_2, x_3, x_4, then we can construct anharmonic ratios/cross ratios, which are invariant under CTs given by -

$$\Gamma(x_1, x_2, x_3, x_4) = \frac{x_{12}x_{34}}{x_{13}x_{24}} \text{ or } \frac{x_{12}x_{34}}{x_{14}x_{23}} \tag{4.69}$$

So that,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{14}x_{23}}\right) \prod_{i<j}^{4} x_{ij}^{\frac{\Delta}{3}-\Delta_i-\Delta_j} \tag{4.70}$$

where, $\Delta = \sum_{i=1}^{4} \Delta_i$.

4.4.4 Ward Identities for Conformal Symmetry

Ward Identity for translation invariance is:

$$\partial_\mu \langle T^\mu_\nu X \rangle = \sum_i \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle \tag{4.71}$$

This also holds true even after modification - as in equation 4.60. (Identical Divergence-less)

Ward Identity for Lorentz Invariance:

$$j^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \tag{4.72}$$

This can again be obtained post-modification using Belifante tensor . . .. Here $T^{\mu\nu}$ is the same as that of translation stress tensor! That's the result of Belifante modification or any symmetrization scheme.

So Ward Identity reads,

$$\partial_\mu \langle (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) X \rangle = \sum_i \delta(x - x_i) \left[\left(x_i^\nu \partial_i^\rho - x_i^\rho \partial_i^\nu \right) \langle X \rangle - i S_i^{\nu\rho} \langle X \rangle \right] \tag{4.73}$$

S_i is the spin generator for the i-th field ϕ_i.

Using 4.71 we can further write this as

$$\langle T^{\rho\nu} - T^{\nu\rho} X \rangle = -i \sum_i \delta(x - x_i) S_i^{\nu\rho} \langle X \rangle . \tag{4.74}$$

$T^{\mu\nu}$ symmetric within correlation functions, except at the contact points!

For scale invariance: we have for $d > 2$ (but we will assume it can be done for d = 2 as well)

$$j_D^\mu = T_\nu^\mu x^\nu$$

from equation 4.62 (T modified to be traceless on the shell).

Ward Identity reads -

$$\partial_\mu \langle T_\nu^\mu x^\nu X \rangle = -\sum_i \delta(x - x_i) \left\{ x_i^\nu \frac{\partial}{\partial x_i^\nu} \langle X \rangle + \Delta_i \langle X \rangle \right\} \tag{4.75}$$

Using equation 4.71 again gives,

$$\langle T_\mu^\mu X \rangle = -\sum_i \delta(x - x_i) \Delta_i \langle X \rangle . \tag{4.76}$$

The stress tensor is traceless again except at the contact points!

Equations 4.71, 4.74, 4.76 are the three ward identities for Conformal Invariance. We have obtained the above correlation functions by substitution of appropriate currents in the general Ward identity B.9.

4.4.5 Tracelessness of the stress tensor in 2 dimensions

Our goal is to *show that vacuum expectation value of the trace of energy-momentum tensor (or of its square) vanishes in 2 dimensions if the theory has scale, rotation and translation invariance!* This is basically a discussion on the lines of whether the stress tensor can be made traceless (thus implying the considered theory is conformally invariant) given scale, Lorentz and translation invariance. We already showed this for $d > 2$. Here, we try to justify our generalization of that to even $d = 2$ by showing that the expectation value of the trace of energy-momentum tensor vanishes between ground states anyway.

We consider a 2-point function of Energy-Momentum Tensor called *Schwinger Function*.

$$S_{\mu\nu\rho\sigma}(x) = \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle \tag{4.77}$$

By translation, scale symmetry and symmetrization, we get

- $$\begin{aligned} S_{\mu\nu\rho\sigma} &= \langle T_{\mu\nu}(0) T_{\rho\sigma}(-x) \rangle \\ &= \langle T_{\rho\sigma}(-x) T_{\mu\nu}(0) \rangle \\ &= S_{\rho\sigma\mu\nu}(-x) \overset{\text{if parity symmetry}}{=} S_{\rho\sigma\mu\nu}(x) \end{aligned}$$

- $S_{\mu\nu\rho\sigma} = S_{\nu\mu\rho\sigma} = S_{\mu\nu\rho\sigma} = S_{\nu\mu\sigma\rho}$

- $S_{\mu\nu\rho\sigma}(\lambda x) = \lambda^{-4} S_{\mu\nu\rho\sigma}$

The general form of the functions given these symmetries can be written as,

$$S_{\mu\nu\rho\sigma}(x) = (x^2)^{-4}\Big\{ A_1 g_{\mu\nu} g_{\rho\sigma}(x^2)^2 + A_2 \left(g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} \right) (x^2)^2 \tag{4.78}$$

$$+ A_3 \left(g_{\mu\nu} x_\rho x_\sigma + g_{\rho\sigma x_\mu x_\nu} \right) x^2 + A_4 x_\mu x_\nu x_\rho x_\sigma \Big\} \tag{4.79}$$

$\partial_\mu T^{\mu\nu} = 0$ extends to the Schwinger function to give

$$S_{\mu\nu\rho\sigma}(x) = \frac{A}{(x^2)^4}\Big\{ \left(3 g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) (x^2)^2 - 4 \left(g_{\mu\nu} x_\rho x_\sigma + g_{\rho\sigma x_\mu x_\nu} \right) x^2 + 8 x_\mu x_\nu x_\rho x_\sigma \Big\} \tag{4.80}$$

And it follows simply that

$$S_{\mu\ \sigma}^{\mu\ \sigma}(x) \equiv \langle T_{\mu}^{\mu}(x) T_{\sigma}^{\sigma}(0) \rangle = 0 \tag{4.81}$$

In particular, we have

$$\langle T_{\mu}^{\mu}(0)^2 \rangle = 0. \tag{4.82}$$

Thus, operator T_{μ}^{μ} has zero expectation value and zero standard deviation in the ground state!

But the general result is equation 4.75 as already noted - the trace of the EM tensor vanishes within correlation functions except at contact points.

Chapter 5

Conformal Invariance in 2 Dimensions

In two dimensions, the symmetry group is extended. In addition to the global symmetries we already noted in the last chapter, there are local symmetries forming an infinite dimensional algebra. These additional transformations are essentially the holomorphic and anti-holomorphic functions on the complex plane. After studying these in detail, we rewrite the symmetry identities in the (anti) holomorphic form along the while motivating the behavior of (conformal) fields given by a conformal dimension. We can then read out the operator product expansion of Stress tensor with other conformal fields. Example cases of free Boson, free Fermion and Ghost system are discussed where we identify the conformal fields and their product expansion with stress tensor. In all the examples, we note the anomalous term in the product expansion of stress tensor with itself deviating from the conformal nature of a field. This is given by the central charge of the theory, and we relate it to the quantum breaking of the conformal symmetry due to introduction of a macroscopic scale. It is a special case of trace anomaly or Weyl anomaly in general dimensions. This is an important aspect of CFT, and we will discuss this further while connecting AdS/CFT and Geometry in Chapter 9.

5.1 Conformal Transformations in 2 Dimensions - Local, Global and The Witt Algebra

Start with coordinates $z^0, z^1 \in \mathbb{R}^2$. For the transformation $z^\mu \to w^\mu(z)$ to be conformal we need

$$g^{\mu\nu} \to g'^{\mu\nu}(w) = \frac{\partial w^\mu}{\partial z^\alpha}\frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta} \overset{conf.}{=} \Lambda(z) g^{\mu\nu}(z) \tag{5.1}$$

In the case of Euclidean metric $g^{\mu\nu} = \delta^{\mu\nu}$ we have,

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \text{ and } \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{5.2}$$

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \text{ and } \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{5.3}$$

These are just the Cauchy-Riemann conditions for the function

$$w(z^0, z^1) = w^0(z^0, z^1) + iw^1(z^0, z^1) \tag{5.4}$$

to be holomorphic or anti-holomorphic.

An equivalent formulation can be drawn by using complex coordinates $z, \bar{z}$ as follows:

$$z = z^0 + iz^1, \qquad \bar{z} = z^0 - iz^1 \tag{5.5}$$

$$z^0 = \frac{1}{2}(z + \bar{z}), \qquad z^1 = \frac{1}{2i}(z - \bar{z}) \tag{5.6}$$

$$\partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \equiv \partial, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \equiv \bar{\partial} \tag{5.7}$$

$$\partial_0 = \partial_z + \partial_{\bar{z}}, \qquad \partial_1 = i(\partial_z - \partial_{\bar{z}}) \tag{5.8}$$

This allows us to write the holomorphic and anti-holomorphic conditions as

$$\partial_{\bar{z}} w(z, \bar{z}) = 0, \ \partial_z \bar{w}(z, \bar{z}) = 0 \tag{5.9}$$

Thus,

$$w(z, \bar{z}) \equiv w(z), \text{ and } \bar{w}(z, \bar{z}) \equiv \bar{w}(\bar{z}) \tag{5.10}$$

The metric in the coordinates $z, \bar{z}$ becomes -

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \; g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{5.11}$$

The metric allows us to move between the covariant holomorphic index and contravariant anti-holomorphic index as follows:

$$g^{\mu\nu} z_{\mu} = z^{\nu}, \; g_{\mu\nu} y^{\mu} = y_{\nu} \tag{5.12}$$

with $z_{\mu} = (z, \bar{z}) \implies z^{\nu} = 2(\bar{z}, z)$ and $y^{\mu} = (\bar{y}, y) \implies y_{\nu} = \frac{1}{2}(y, \bar{y})$. Basically, the metric swaps the hol. and anti-hol. index but also adds a factor of 2.

We also write the anti-symmetric matrix $\varepsilon_{\mu\nu}$ as follows:

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix}, \; \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \tag{5.13}$$

We have basically let $\varepsilon_{\mu\nu}^{\text{hol.}} = \varepsilon_{\mu\nu}^{cart.} \times J$. Where J is the Jacobian.

The above discussion on transformations is summarized as follows -

$$z, \bar{z} \longrightarrow w(z), \bar{w}(\bar{z}) \tag{5.14}$$

- The proper way to think about this in the light of real coordinates (z^0, z^1), which we began with, is to let $z^0, z^1 \in \mathbb{C}$ so that equations 5.5-5.8 just correspond to change of independent variables. And to get back the real coordinates z^0, z^1, we consider the real surface $z^* = \bar{z}$. This is our physical space. We continue to make deductions on various forms of correlators or other functions based on this scheme of considering the real surface underneath. This offers a convenient representation of the theory in the holomorphic and anti-holomorphic forms, as we will see - they decouple.

- Any analytical map between complex planes is conformal! One can trivially see this by -

$$dw = \frac{\partial w}{\partial z} dz + \cancel{\frac{\partial w}{\partial \bar{z}}} d\bar{z} = \frac{dw}{dz} dz \tag{5.15}$$

The dilation factor is given by the $| \frac{dw}{dz} |$ and the rotation by Arg$(\frac{dw}{dz})$.

So conformal transformations in 2 dimensions are the set of all analytical maps, with group multiplication being the composition of maps. Since such functions admit the Laurent series, which contains infinite parameters/coefficients - this class of 2d transformations is infinite-dimensional. This becomes clearer when we derive the algebra of generators of such transformations.

- Although it might look like we are talking about coordinates $(z, \bar{z})$ on $\mathbb{C}^2$, we will mostly just deal with holomorphic functions and anti-holomorphic functions on $\mathbb{C}$ because the whole theory decouples into these, just like the transformations already did too! Most of the time, we will also not discuss the anti-holomorphic part of the theory since it runs parallelly, and we can always add it back trivially most of the time.

- Actually we work not on $\mathbb{C}$ but on the Riemann sphere $\mathbb{C} \cup \infty$ (identifying the point at infinity with the tip of the sphere) which is compact. That is while we are moving from R^2 to $\mathbb{C}$ we are also identifying infinity with a single point and compactifying it. The reason for doing this here might be relevant to the fact that poles become removable singularities on the range $\mathbb{C} \cup \infty$. That is, holomorphic functions are then allowed to have poles too! However, note that such compact spaces are more physically helpful due to easier boundary conditions. We thus have dual benefits of such a compactification - trivial boundary conditions and a bigger class of transformations.

This information should allow us to see the behaviour of fields under such transformations.Any holomorphic infinitesimal transformation using the Laurent expansion around $z = 0, \bar{z} = 0$ can be written as -

$$z' = z + \epsilon(z);\ \epsilon(z) = \sum_{\infty}^{+\infty} c_n z^{n+1}. \tag{5.16}$$

and similarly $\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$

So, for a spinless field, the variation in the field is given by -

$$\begin{aligned}\phi'(z', \bar{z}') &= \phi(z, \bar{z}) \\ &= \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}'(\bar{z}')\bar{\partial}'\phi(z', \bar{z}')\end{aligned}$$

So,

$$\begin{aligned}\partial\phi &= -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi && (5.17)\\ &= \sum_n \left\{c_n l_n \phi(z, \bar{z}) + \bar{c}_n \bar{l}_n \phi(z, \bar{z})\right\} && (5.18)\end{aligned}$$

where,

$$l_n = -z^{n+1}\partial_z,\ \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{5.19}$$

are the generators of the local conformal transformations acting on the fields, akin to A.21. We can derive the algebra of these generators to be -

$$[l_n, l_m] = (n - m)\, L_{n+m} \tag{5.20}$$

$$\left[\bar{l}_n, \bar{l}_m\right] = (n - m)\, \bar{l}_{n+m} \tag{5.21}$$

$$\left[l_n, \bar{l}_m\right] = 0 \tag{5.22}$$

We deduce that the local conformal algebra is a direct sum of two isomorphic algebras: $\mathcal{L} \oplus \bar{\mathcal{L}}$. This is called the **Witt Algebra**, which is clearly infinite-dimensional.We are calling the generators and thus the algebra *local* because they are not defined everywhere, and in fact, such transformations do not exactly form a group since they are not all defined and invertible everywhere. We can look for a subset of these transformations, which are *global*, those which are well behaved as $z \longrightarrow 0$ and $z \longrightarrow \infty$. [$z \longrightarrow \infty \equiv w \longrightarrow 0,\ z = -\frac{1}{w}$]

Such well behavedness condition on 5.16 gives,

$$c_n = 0 \forall n < -1,\ c_n = 0 \forall n > 1. \tag{5.23}$$

So we have $\{l_{-1}, l_0, l_1\} \cup \left\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\right\}$ generating the global conformal transformations.

$$l_{-1} = -\partial_z,\ l_0 = -z\partial_z,\ l_1 = -z^2\partial_z. \tag{5.24}$$

At this point, we can compare these with table 4.1 applied to 2 dimensions.

- l_{-1} and $\bar{l}_{-1}$ generate translations.
- $l_0 + \bar{l}_0$ generates dilations.
- $i(l_0 - \bar{l}_0)$ generates rotations.
- $l_1, \bar{l}_1$ generate special conformal transformations.

  Convert the differential generators to Cartesian versions to immediately see that these indeed are the rotation and translations.

These are the *true* conformal generators leading to the global conformal group or *special conformal group* akin to the conformal groups in other dimensions. But in 2 dimensions, we have a larger class of conformal transformations possible, albeit local. Generators preserving the real surface are $l_n + \bar{l}_n$ and $i\left(l_n - \bar{l}_n\right)$ which corresponding to dilations and rotations for $n = 0$.

We can derive the finite form of the global transformation from the infinitesimal transformation,

$$z' = z + c_{-1} + c_0 z + c_1 z^2 \equiv z + \epsilon \left(a z^2 + b z + c \right) \tag{5.25}$$

$$\dot{z} = a z^2 + b z + c \tag{5.26}$$

The key steps for the integration are highlighted below. Let r_1, r_2 denote the roots of a quadratic expression $ax^2 + bx + c$

$$\begin{aligned} \dot{z}\left(\frac{1}{z+r_1} - \frac{1}{z+r_2}\right) &= \sqrt{b^2 - 4ac} \\ \ln\left(\left|\frac{z'+r_1}{z'+r_2}\right|\left|\frac{z+r_2}{z+r_1}\right|\right) &= \sqrt{b^2-4ac}\,\epsilon' \\ \frac{z'+r_1}{z'+r_2} &= k\frac{z+r_1}{z+r_2},\ \text{choose } \epsilon' = \ln k/\sqrt{b^2-4ac} \\ z' &= \frac{z\left(r_1 - kr_2\right) + (1-k)\, r_1 r_2}{(k-1)\, z + (kr_1 - kr_2)} \end{aligned} \tag{5.27}$$

Since a, b, c were arbitrary, we may choose to start with $ax^2 + \frac{(A-D)}{2}ax - \frac{B}{C}a$ and get the 2d finite form of global conformal transformation to be,

$$z' = \frac{Az + B}{Cz + D} \tag{5.28}$$

for some arbitrary $A, B, C, D \in \mathbb{C}$. So, the set of global transformations in 2D (forming a *Special Conformal Group*) is given by the functions

$$f(z) = \frac{az + b}{cz + d} \text{ s.t. } ad - bc = 1 \text{ for } a, b, c \in \mathbb{Z} \tag{5.29}$$

These are called 'Projective Transformations', isomorphic to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}!$ And $SL(2, \mathbb{C}) \cong S)(3, 1)!$ So, the global conformal group in 2D is just a 6-parameter (3 complex) group

$SO(3,1)$. We can adopt another direction to prove this form of global transformation. By demanding properties $f(z)$ must satisfy, thereby restricting it.

1. f **shouldn't have any branch point**. Because uniqueness shall not hold around it! Can't we define branch cuts? Branch cuts essentially reflect the non-isolated nature of such singularities (branch points). To remove these, we must modify our domain to a Riemann Surface.

2. f **shouldn't have any essential singularities.** Holomorphic functions around these points behave wildly! Casorati-Weirstrass theorem says that, around an essential singularity, a holomorphic map takes all the values of $\mathbb{C}$ in **any** arbitrary small neighbourhood! These are strictly non-removable since it is not bounded around the singularity. This implies non-invertibility around that singularity.

3. Since we have already extended our range to the Riemann Sphere, i.e. compactify the Complex plane by identifying the point at infinity, poles become removable singularities. This means functions with poles are also possible holomorphic functions (such functions are called meromorphic). We also avoid the isolated removable singularities so that our function can be defined in one piece.

$$f(z) = \frac{P(z)}{Q(z)} \tag{5.30}$$

   This is the ratio of polynomials with no common zeroes. Any holomorphic function with poles being the only singularities is written in the above form.

   (a) $P(z)$ can't have multiple zeroes - for invertibility at zero.

   (b) If $P(z)$ has multiple zero z_0 of order $n > 1$ then the inverse has a branch point.

   So,

$$P_(z) = az + b,\ a, b \in \mathbb{C}. \tag{5.31}$$

4. Q(z)?

   (a) Q(z) can't have multiple zeroes again for the same reason.

   (b) Q(z) can't have zero of order $n > 1$ again.

   So, Q(z) = cz + d.

5. $f_1 \circ f_2 = A_1 A_2$. This can be trivially shown.

6. $f_1 \circ f_2(z) = z \iff A_1 A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \mid A_1 \mid \neq 0, \mid A_2 \mid \neq 0$ $\iff a_1 d_1 - b_1 c_1 \neq 0, a_2 d_2 - b_2 c_2 \neq 0$. Thus, we can choose the normalization $ad - bc = 1$, since $f(z)$ is invariant under overall scaling of a, b, c, d!

Now that we understand the coordinate transformations, let's look at how fields behave under these.

5.2 Primary Fields

We already know how fields behave under the true/global conformal group from the previous section. But in 2 dimensions, we also have local conformal transformations. Since $l_0 + \bar{l}_0$ generates dilations, we expect $l_0 + \bar{l}_0 = -i\Delta$. It is simpler to consider the spin fields too in 2 dimensions - the representation of Lorentz(spin) algebra is just $S_{\mu\nu} = s\varepsilon_{\mu\nu}$ (in 2d, we just have a single independent parameter to be generated via the basis of anti-symmetric matrices - thus appears a *planar spin*). We thus expect $i(l_0 - \bar{l}_0) = -i\frac{1}{2}\Omega^{\mu\nu}S_{\mu\nu} \equiv s$ (we use the definition 5.13)

The $l_0, \bar{l}_0$ can be thus deduced from above,

$$\begin{aligned} l_0 &= \left(\frac{1}{2}\left(l_0 + \bar{l}_0\right) - \frac{i}{2}\left[i\left(l_0 - \bar{l}_0\right)\right]\right)\phi \\ &= \left(-\frac{i}{2}\Delta - \frac{i}{2}s\right) \\ &= -\frac{i}{2}\left(\Delta + s\right) \qquad (5.32) \\ \bar{l}_0 &= -\frac{i}{2}\left(\Delta - s\right) \qquad (5.33) \end{aligned}$$

Thus, we are motivated to define the conformal weights,

$$h = \frac{1}{2}\left(\Delta + s\right), \; \bar{h} = \frac{1}{2}\left(\Delta - s\right) \tag{5.34}$$

1. We note that in 2 dimensions, the scale and spin are placed on an equal footing - captured in the conformal weights $h, \bar{h}$, which appear in l_0 and $\bar{l}_0$ respectively.

2. These conformal weights allow us to place the above transformations on the same lines of 4.27. To do this, first note that,

   $$g'_{\mu\nu}(w, \bar{w}) = \left(\frac{dz}{dw}\right)\left(\frac{d\bar{z}}{d\bar{w}}\right) g_{\mu\nu}(z, \bar{z}) \tag{5.35}$$

   This tells us that the conformal scaling factors appear decoupled into holomorphic and anti-holomorphic parts. A straightforward extrapolation of the transformation law of quasi primaries would include the conformal weight instead of just the scale dimensions and include both holomorphic and anti-holomorphic contributions, i.e.,

   $$\phi'(w, \bar{w}) = \left(\frac{dz}{dw}\right)^{h}\left(\frac{d\bar{z}}{d\bar{w}}\right)^{\bar{h}} \phi(z, \bar{z}) \tag{5.36}$$

   Note that in two dimensions, Quasi-Primaries transform as

   $$\phi'(x') = \Lambda(x)^{\frac{\Delta}{2}}\phi(x) \tag{5.37}$$

So, the holomorphic form of transformation law of Quasi-Primary spin full fields in 2D is

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) \tag{5.38}$$

Under the general class of transformations 5.31, we now know the behaviour under the subclass, viz., true conformal group. We generalize the transformation law 5.38 beyond this subclass to the whole of 5.31. Not all fields may behave in that manner, but the ones that do, we will call them *Primary Fields*. That is, Primary fields are those that behave like 5.38 not just under global conformal transformations but also under local ones. These are generalizations of Quasi-Primaries to local transformations.

Much of the discussion in the 2D CFT revolves around the insights we gain by doing so. We will use that form as if it's true for the large class and derive stronger results with weaker sub-cases for the global transformations, which are the true symmetries. Then, we will physically link the results obtained. The next few sections will give the results leading up to the central charge - the conformal anomaly, which captures the breaking of local conformal symmetry in the Quantum picture. Also, we call any field that is not primary as secondary.

Note that under infinitesimal transformations $w = z+\epsilon(z), \bar{w} = \bar{z}+\bar{\epsilon}(\bar{z})$ transformation law 5.38 becomes,

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \phi'(z,\bar{z}) - \phi(z,\bar{z}) \tag{5.39}$$

$$= -\left(h\phi\partial_z\epsilon + \epsilon\partial_z\phi\right) - \left(\bar{h}\phi\partial_Z\bar{\epsilon} + \bar{\epsilon}\partial_z\phi\right) \tag{5.40}$$

5.3 Holomorphic form of Correlation functions, and Ward Identities

Let's rewrite the correlation functions for 2 dimensions in the above form. Recall that if $n >$ 4, we can construct conformally invariant cross-ratios. In 2D, with four points z_1, z_2, z_3, z_4

we only have a single independent cross-ratio due to co-planarity.

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}},\ 1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}},\ \frac{\eta}{1-\eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}} \tag{5.41}$$

Under (local) conformal transformations, the correlator of primary fields (n of them) ϕ_i with conformal dimensions $h_i, \bar{h}_i$ transformations as

$$\langle \phi_1(w_1,\bar{w_1}),\ldots,\phi_n(w_n,\bar{w}_n)\rangle = \prod \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle \phi_1(z_1,\bar{z}_1),\ldots,\phi_n(z_n,\bar{z}_n)\rangle \tag{5.42}$$

For rotation invariance, we must have any correlation function to be dependent on $(z_{ij}\bar{z}_{ij})^{1/2}$, so that on the real surface this is just $|z_{ij}|$.

The holomorphic form of the correlators then is straightforward to write,

$$\langle \phi_1(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \tag{5.43}$$

$$\begin{aligned}\langle \phi_1(z_1,\bar{z}_z),\phi_2(z_2,\bar{z}_2),\phi_3(z_3,\bar{z}_3)\rangle &= C_{12}\frac{1}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}} \\ &\times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\end{aligned} \tag{5.44}$$

$$\langle \phi_1(z_1,\bar{z}_z),\phi_2(z_2,\bar{z}_2),\phi_3(z_3,\bar{z}_3),\phi_4(z_4,\bar{z}_4)\rangle = f(\eta,\bar{\eta})\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \tag{5.45}$$

where, $h=\sum h_i, \bar{h}=\sum\bar{h}_i$.

The novelty here in 2D is that a non-zero spin is also included $(h_i-\bar{h}_i)$!

The Ward Identities for the conformal invariance (corresponding to translation, rotation and scaling) are,

$$\frac{\partial}{\partial x^\mu} \langle T^\mu_\nu(x) X \rangle = -\sum_i^n \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle \tag{5.46}$$

$$\varepsilon \langle T^{\mu\nu}(x) X \rangle = -i \sum_i^n s_i \delta(x - x_i) \langle X \rangle \tag{5.47}$$

$$\langle T^\mu_\mu(x) X \rangle = -\sum_i^n \delta(x - x_i) \Delta_i \langle X \rangle \tag{5.48}$$

where we have used the two-dimensional representation of the spin operator S.

Classically, $T^{\mu\nu}$ is a symmetric and traceless Energy-Momentum Tensor. And in the Quantum picture, it remains to be so, except at the contact terms. The conservation laws of the translation, rotation, and scaling can be placed on a connected footing as follows - which helped us to write the ward identities in terms of the stress tensor.

Symmetry	Conservation Law	Canonical Current (obtained via A.35)	Modified Current
Translation	$\partial_\mu T^{\mu\nu} = 0$	$T_c^{\mu\nu}$	$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu}$
Rotation	$\partial_\mu j^{\mu\nu\rho} = 0$	$j_c^{\mu\nu\rho}$	$j^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$
Scaling	$\partial_\mu j^\mu = 0$	j_c^μ	$j_D^\mu = T^\mu_\nu x^\nu$

Table 5.1: Conserved Currents for different symmetries along with their modified forms in terms of a symmetric and traceless stress tensor.

We wish to write the above ward identities in holomorphic form. The first step would be to deduce a delta function that we can integrate over holomorphic or anti-holomorphic functions.

- Gauss Divergence theorem to convert an integral on $\mathbb{C}^2$ to C.
- Cauchy's Theorem for holomorphic or anti-holomorphic functions to get the residue as the function at $x = 0$ or perhaps hol(x) $= z = 0$.

Claim:

$$\delta(x) = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}} \tag{5.49}$$

The first(second) form can be used for holomorphic (anti) functions. $x \equiv (z, \bar{z})$.

Proof:

Note that: $\int_{M \subset \mathbb{C}^2} d^2x \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu$ for a vector F^μ.

Here, $d\xi_\mu = \varepsilon_{\mu\rho} ds^\rho$ is the outward directed differential.

So,

$$\begin{aligned}\int_M d^2x \partial_\mu F^\mu &= \int_{\partial M} dz \varepsilon_{\bar{z}z} F^{\bar{z}} + d\bar{z} \varepsilon_{z\bar{z}} F^z \\ &= \frac{1}{2} i \int_{\partial M} \left\{ -dz F^{\bar{z}} + d\bar{z} F^z \right\}\end{aligned} \tag{5.50}$$

Now this is ripe for Cauchy's Theorem -

$$\begin{aligned}\int_M d^2x \delta(x) f(z) &= \frac{1}{\pi} \int_M d^2x f(z) \partial_{\bar{z}} \frac{1}{z} \\ &= \frac{1}{\pi} \int_M d^2x \partial_{\bar{z}} \left(\frac{f(z)}{z} \right) \\ &= \frac{1}{2\pi i} \oint_{\partial M} dz \frac{f(z)}{z} \\ &= f(0)\end{aligned} \tag{5.51}$$

This is actually brilliant! Basically, the derivative w.r.t the anti-holomorphic coordinate allows us to convert the integral into an integral of the divergence of a holomorphic function - which then becomes a contour integral due to Gauss' Theorem and leaves out a residue due to Cauchy's. Similarly, we have the other case - integrating over an anti-holomorphic function $\bar{f}(\bar{z})$ gives $\bar{f}(0)$.

Using this in the Cartesian ward identities and expanding out the tensor transformation law, for a change of variables - $(x^0, x^1) \longrightarrow (x'^0, x'^1) \equiv (z = x^0 + ix^1, \bar{z} = x^0 - ix^1)$ gives the holomorphic form of ward identities,

$$2\pi\partial_z \langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}} \langle T_{zz}X\rangle = -\sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z-w_i}\partial_{w_i} \langle X\rangle \tag{5.52}$$

$$2\pi\partial_z \langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}} \langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n} \partial_{z} \frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i} \langle X\rangle \tag{5.53}$$

$$2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n} \delta(x-x_i)\Delta_i \langle X\rangle \tag{5.54}$$

$$-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n} \delta(x-x_i)s_i \langle X\rangle \tag{5.55}$$

Add and subtract the last two equations to get,

$$2\pi \langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z-w_i}h_i \langle X\rangle \tag{5.56}$$

$$2\pi \langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n} \partial_{z} \frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i \langle X\rangle \tag{5.57}$$

Subbing these in the first two equations gives,

$$\partial_{\bar{z}}\left\{\langle T(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_1}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right]\right\} = 0 \tag{5.58}$$

$$\partial_{z}\left\{\langle \bar{T}(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_1}\partial_{\bar{w}_i}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle X\rangle\right]\right\} = 0 \tag{5.59}$$

Where, $T = -2\pi T_{zz}$, $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$.

So we have a holomorphic (anti) condition on the function in equation 5.58 (5.59).To avoid any additional singular terms in the stress tensor apart from those which appear in the ward identities 5.52-5.55, the above functions must be regular at $z = w_i$

That is, the stress tensor is determined within correlation functions up to regular terms (at contact points) as follows -

$$\langle T(z)X\rangle = \sum_{w=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right\} + reg. \tag{5.60}$$

Similarly, for the anti-holomorphic part of the stress tensor.

Note that the mixed components $T_{z\bar{z}}, T_{\bar{z},z}$ as given by 5.56,5.57 are just δ functions ready to act on some holomorphic or anti-holomorphic functions and spitting out its details near contract terms.

Note that ward identities make sense only as distributions irrespective of cartesian/holomorphic forms.

What we have done so far doesn't talk to the local transformations yet. Since all we have done is rewrite the global ward identities in 2 dimensions, including spin. $h, \bar{h}$ appear due to a non-zero spin.

5.3.1 Conformal Ward Identity

There is a slick way to derive a ward identity that actually talks to the 'local transformations'!

First, note that,

$$\partial_\mu \left(\epsilon_\nu T^{\mu\nu}\right) = \epsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2}\left(\partial_\rho \epsilon^\rho\right) \eta_{\mu\nu} T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta \varepsilon_{\mu\nu} T^{\mu\nu} \tag{5.61}$$

This is because,

$$\frac{1}{2}\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) = \frac{1}{2}(\partial_\rho \epsilon^\rho)\eta_{\mu\nu} \tag{5.62}$$

$$\frac{1}{2}\left(\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu\right) = \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta \varepsilon_{\mu\nu} \tag{5.63}$$

the first one is due to conformal transformations, and the second is just identity.

Using this within the correlation function, integrating and using the ward identities for scaling, translation, and rotation gives,

$$\int d^2x \left\langle \partial_\mu \left[\epsilon_\nu T^{\mu\nu} \right] X \right\rangle = \int d^2x \left\{ \epsilon_\nu \left\langle \partial_\mu T^{\mu\nu} \right\rangle + \frac{1}{2}\partial \cdot \epsilon \left\langle T^\mu_\mu \right\rangle + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta \left\langle \varepsilon_{\mu\nu} T^{\mu\nu} X \right\rangle \right\} \tag{5.64}$$

$$= -\sum_{i=1}^{n} \int d^2x \delta(x - x_i) \left\{ \epsilon^\nu \partial_{i\nu} + \frac{1}{2}\partial \cdot \epsilon \Delta_i + \frac{i}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta s_i \right\} \langle X \rangle \tag{5.65}$$

$$= -\sum_{i=1}^{n} \int d^2x \delta(x - x_i) \left\{ \epsilon \partial_i + h_i \partial \epsilon + \epsilon \bar{\epsilon} \bar{\partial}_i + \bar{h}_i \bar{\partial} \bar{\epsilon} \right\} \langle X \rangle \tag{5.66}$$

We recognize that r.h.s of above equation is just $\delta_{\epsilon\bar{\epsilon}} \langle X \rangle$From equation 5.39.

So,

$$\delta_{\epsilon\bar{\epsilon}} \langle X \rangle = \int_M d^2x \partial_\mu \left\langle T^{\mu\nu}(x) \epsilon_\nu(x) X \right\rangle \tag{5.67}$$

Now, use the Gauss' Law with $F^\mu = \langle T^{\mu\nu}(x)\epsilon_\nu(x) X \rangle$

$$\delta_{\epsilon\bar{\epsilon}} \langle X \rangle = \frac{1}{2} i \int_C \left\{ -dz \left\langle T^{\bar{z}\bar{z}} \epsilon_{\bar{z}} X \right\rangle + d\bar{z} \left\langle T^{zz} \epsilon_z X \right\rangle - \cancel{ds \left\langle T^{\bar{z}z} \epsilon_z X \right\rangle} + \cancel{d\bar{z} \left\langle T^{z\bar{z}} \epsilon_{\bar{z}} X \right\rangle} \right\} \tag{5.68}$$

We have $T^{\bar{z}z} = T_{z\bar{z}}$ and $T^{z\bar{z}} = T_{\bar{z}z}$ which are basically δ functions appearing in the integrals over the wrong variables, which makes both the terms vanish!

Thus we have the Conformal Ward Identity (integral form) as

$$\delta_{\epsilon\bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \tag{5.69}$$

This is just the holomorphic form of the Ward Identities for Quasi-Primary Fields in a compact form!

Some key results about the conformal ward identity -

1. Indeed, the local transformation law of primary fields can be reproduced.

2. For global transformations, this gives $\partial_{\epsilon\bar{\epsilon}} = 0$ as it should be! The infinitesimal form of the projective transformation is

$$f(z) = \frac{(1+\alpha)z + \beta}{\gamma z + (1-\alpha)} \tag{5.70}$$

$$\epsilon(z) = \beta + 2\alpha z - \gamma z^2 + \ldots \tag{5.71}$$

The R.H.S of 5.69 should vanish for above $\epsilon(z)$. Then, R.H.S of equation 5.66 should vanish, giving the following relations on correlators of Primary fields,

$$\sum_i \partial_{w_i} \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0 \tag{5.72}$$

$$\sum_i \left(w_i \partial_{w_i} + h_i\right) \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0 \tag{5.73}$$

$$\sum_i \left(w_i^2 \partial_{w_i} + 2w_i h_i\right) \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0 \tag{5.74}$$

The correlation functions 5.42 satisfy the above relations! And in fact, there's an $\Longleftrightarrow$ relation between the form of correlation functions and the above relations.

We note the following important steps -

1. The Global sub-algebra is captured in Witt Algebra - which is the local conformal algebra. Any calculation done using local algebra is also valid on the global algebra. So we just do that and aim to see if we can get anything more than the global algebra results.

2. The Ward Identities of Global Symmetry is captured in Conformal Ward Identity - which we will take to be valid for **all** the fields. Note that we haven't used anywhere that the local transformations are symmetries. However, we have derived the identity using the transformation law of quasi-primary fields. We could extend this to all the fields satisfying the same transformation law (under local transformations), i.e. Primary fields. However, this identity is generally regarded as the definition of the effect of local conformal transformations of any field. Let's see what we get by doing so!

So, the conformal ward identity tells us the effect of local conformal symmetry (since we assumed it to be valid for primary fields, too). It also fixes the form of correlation functions via global transformations.

5.3.2 Behavior of T(z):

We want the energy-momentum tensor to be well defined everywhere $\implies T(0)$ should be finite.

Claim: $T(z)$ should decay as z^{-4} as $z \to \infty$.

Proof 1: EM Tensor has scaling dimension 2, and spin dimension as 2 $\implies h = 2, \bar{h} = 2$. Under $z \to w = \frac{1}{z}$ (a global conformal transformation),

$$T'(w) = \frac{dw}{dz}^{-2} T(z) = z^4 T(z)$$

So we must have as $z \to \infty$, $T(z) \to z^{-4}$ So that $T'(\frac{1}{z})$ is finite.

Proof 2: Consider $\delta_\epsilon \langle 1 \rangle$ which must vanish under Global Conformal transformation,

$$\implies -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) \rangle = 0$$

Since $\epsilon(z)$ is quadratic for SCTs, and the above relation should be true for the contour circling infinity, $T(z)$ must behave as z^{-4} near infinity if no reside to be picked up, so that $\oint = 0$!

5.4 Operator Product Expansions

Inspired by computations in the last section (and probably many more such calculations), we represent the product of operators within the correlation functions by a sum expansion with each term consisting of c-number function of $z - w$, which may diverge at $z = w$ multiplied to an operator which is non-singular at $z = w$.

Note that the divergences in the correlation function at the contact terms reflect the infinite fluctuation of quantum fields at a precise position. Even the average of a field over a point diverges, too. [$\phi_{av.} = \frac{1}{V}\int_V d^2x\phi(x)$ diverges as $V \to \infty$.]

Recall equation 5.59. This is a form of the expansions we are talking about. For a single primary field $\phi(x)$, this becomes,

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w,\bar{w}) \tag{5.75}$$

similarly, the expansion for the product with the anti-holomorphic component of the stress tensor. This expansion is valid up to regular terms. (Since Ward Identities do not fix the regular terms.)

In general, we write the OPE of two fields $A(z)B(w)$ as

$$A(z)B(w) \sim \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{5.76}$$

where the composite fields $\{AB\}_n(w)$ are non-singular at $w = z$. For e.g. $\{T\phi\} = \partial_w\phi(w)$.

Actually, these are just fields as of now - Operator Formalism will come soon, and we will use all this over there. (Next Chapter). We are thus now aware of the operator product structure (between a stress tensor and a primary field of conformal dimension $(h,\bar{h})$)within a correlation function for an arbitrary 2d CFT. Let's see this in a few basic examples admitting scale invariance (thus assumed to be conformally invariant) theories.

Also, note that we never actually derived the scaling ward identity valid for 2d; we just assumed the same $d > 2$ one is also valid here. There is a way to derive the ward identity specifically for 2D.

5.4.1 The Free Boson

Start with the simplest CFT - free, massless, scalar boson ϕ.

$$S = \frac{1}{2} g \int d^2x \partial_\mu \phi \partial^\mu \phi \tag{5.77}$$

where g is the normalization parameter.

Let's first note the procedure for computing the correlation functions in the language of Path Integrals. Assume the theory to be massive. The action can be written as follows,

$$S = \frac{1}{2} \int d^2x d^2y \phi(x) A(x, y) \phi(y) \tag{5.78}$$

with $A(x, y) = g\delta(x - y)\left(\partial^2 + m^2\right)$. This follows from a continuous generalization of the calculation of moments using Gaussian Integrals.

Then, the 2-point correlator is given by $\langle\phi(x)\phi(y)\rangle \equiv K(x, y) = A^{-1}(x, y)$. This can be rewritten as $g(-\partial_x^2 + m^2)K(x, y) = \delta(x - y)$. This is just the Green function for the equation of motion. [We are already familiar with correlation functions being just the green functions of the EOM!]

Rotation and Translation Invariance allow us to directly write $K(x, y) = K(|x - y|) \equiv K(r)$. Integrating the above gives,

$$\begin{aligned} 1 &= 2\pi g \int_0^r d\rho \rho \left(-\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho K'(\rho)) + m^2 K(\rho)\right) \\ &= 2\pi g \left\{-rK'(r) + m^2 \int_0^r d\rho \rho K(\rho)\right\} \end{aligned} \tag{5.79}$$

For $m = 0$, this gives $K(r) = -\frac{1}{2\pi g} \ln r$ -

$$\langle\phi(x)\phi(y)\rangle = -\frac{1}{4\pi g} \ln(x - y)^2 + const. \tag{5.80}$$

And for $m \neq 0$, differentiate equation 5.79 to get the modified Bessel Equation -

$$K''(r) + \frac{1}{r}K'(r) - m^2 K(r) = 0 \tag{5.81}$$

We are interested in the solutions of the above equations that decay near infinity; this allows us to write,

$$K(r) = \frac{1}{2\pi g} K_\circ(mr) \tag{5.82}$$

where,

$$K_\circ(x) = \int_0^\infty dt \frac{\cos xt}{t^2+1};\ (x > 0) \tag{5.83}$$

At large r, $(mr \gg 1)$ $K(r) \sim e^{-mr}$ which is a generic feature of a massive field's correlation functions to decay over the characteristic length or the correlation length of $\frac{1}{m}$. This scale enters into the theory, making massive theories not scale invariant!

In holomorphic coordinates,

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\left\{\ln(z-w) + \ln(\bar{z}-\bar{w})\right\} + const. \tag{5.84}$$

Which gives,

$$\langle \partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \tag{5.85}$$

$$\langle \partial_{\bar{z}}\phi(z,\bar{z})\partial_{\bar{w}}\phi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\frac{1}{(\bar{z}-\bar{w})^2} \tag{5.86}$$

$$\tag{5.87}$$

Let's focus on the holomorphic part of the above,

$$\partial\phi(z)\partial\phi(w) \sim= \frac{1}{4\pi g}\frac{1}{(z-w)^2} \tag{5.88}$$

Recall that, $T_{\mu\nu} = g\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right)$,

So, in complex coordinates, according to discussion around 5.57 -

$$T(z) = -2\pi g : \partial\phi\partial\phi : \tag{5.89}$$

We have also gone ahead and normal ordered it for a vanishing expectation value in the ground state. Normal ordering can be achieved via the following regularization procedure,

$$T(z) = -2\pi g \lim_{w \to z} \left(\partial\phi(z)\partial\phi(w) - \langle \partial\phi(z)\partial\phi(w) \rangle \right) \tag{5.90}$$

$$\begin{aligned} T(z)\partial\phi(w) &= -4\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\ &= \frac{\partial\phi(z)}{(z-w)^2} \end{aligned}$$

Expanding $\partial\phi(z)$ around $z = w$ gives,

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)} \tag{5.91}$$

Compare this with equation 5.75 to conclude that $\partial\phi$ is a primary field with conformal dimension $h = 1$!

Note the OPE between two (holomorphic) stress tensors.

$$\begin{aligned} T(z)T(w) &= 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : n \\ &= \frac{\frac{1}{2}}{(z-w)^4} - \frac{4\pi g : \partial\phi(z)\partial\phi(w) :}{(z-w)^2} \\ &\sim \frac{\frac{1}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned} \tag{5.92}$$

We see that $T(z)$ is not a primary field with conformal dimension $h = 2$. There's an anomalous term proportional to $\frac{1}{(z-w)^4}$!

5.4.2 The Free Fermion

The two-dimensional Euclidean action for a free Majorana fermion is,

$$S = \frac{1}{2} g \int d^2 x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \tag{5.93}$$

where γ^μ are the representations of Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We use the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.94}$$

for $\eta^{\mu\nu} = diag(1,1)$. Note that there is no usual i in the action because we have used $t \to -it$, and the i in γ^1 also reflects the Euclidean representation.

So,

$$\gamma^0 \left(\gamma^0 \partial_0 + \gamma^1 \partial_1 \right) = 2 \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \tag{5.95}$$

Expanding the action by writing the two-component spinor $\Psi \equiv \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$, $\Psi^\dagger \equiv \begin{pmatrix} \psi & \bar{\psi} \end{pmatrix}$ gives,

$$S = g \int d^2 x (\bar{\psi} \partial \bar{\psi} + \psi \bar{\partial} \psi) \tag{5.96}$$

Note that the equations of motion are - $\partial \bar{\psi} = 0$, $\bar{\partial} \psi = 0$ whose solutions are just holomorphic and anti-holomorhpic functions $\bar{\psi}(\bar{z}), \psi(z)$. Now, as before, one can rewrite the action as (so that we can make use of the Gaussian Integrals while calculating the correlation functions) -

$$S = \frac{1}{2} \int d^2 x d^2 y \Psi_i(x) A_{ij}(x,y) \Psi_j(y) \tag{5.97}$$

where, $A_{ij}(x,y) = g\delta(x-y) \left(\gamma^0 \gamma^\mu \right)_{ij} \partial_\mu$.

Then the two-point function is given by,

$$K_{ij}(x, y) = A^{-1}_{ij}(x, y) \tag{5.98}$$

or,

$$g\left(\gamma^0\gamma^\mu\right)_{ik}\frac{\partial}{\partial x^\mu}K_{kj}(x, y) = \delta(x-y)\delta_{ij} \tag{5.99}$$

which gives,

$$2g\begin{pmatrix}\partial_{\bar{z}} & 0\\ 0 & \partial_z\end{pmatrix}\begin{pmatrix}\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle & \langle\psi(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle\\ \langle\bar{\psi}(z,\bar{z})\psi(w,\bar{w})\rangle & \langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle\end{pmatrix} = \frac{1}{\pi}\begin{pmatrix}\partial_{\bar{z}}\frac{1}{z-w} & 0\\ 0 & \partial_z\frac{1}{\bar{z}-\bar{w}}\end{pmatrix} \tag{5.100}$$

$$\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle = \frac{1}{2\pi g}\frac{1}{z-w} \tag{5.101}$$

$$\langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle = \frac{1}{2\pi g}\frac{1}{\bar{z}-\bar{w}} \tag{5.102}$$

$$\langle\psi\ (z,\bar{z})\ \bar{\psi}\ (w,\bar{w})\rangle = 0 \tag{5.103}$$

This further gives,

$$\left\langle\partial_z\psi(z,\bar{z})\psi(w,\bar{w})\right\rangle = -\frac{1}{2\pi g}\frac{1}{(z-w)^2} \tag{5.104}$$

$$\left\langle\partial_z\psi(z,\bar{z})\partial_w\psi(w,\bar{w})\right\rangle = -\frac{1}{\pi g}\frac{1}{(z-w)^3} \tag{5.105}$$

Let's calculate the stress tensor as -

$$T^{\bar{z}\bar{z}} = 2\frac{\partial\mathcal{L}}{\partial\bar{\partial}\Phi}\partial\Phi = 2g\psi\partial\psi \tag{5.106}$$

$$T^{zz} = 2\frac{\partial\mathcal{L}}{\partial\partial\Phi}\bar{\partial}\Phi = 2g\bar{\psi}\bar{\partial}\bar{\psi} \tag{5.107}$$

$$T^{z\bar{z}} = 2\frac{\partial\mathcal{L}}{\partial\partial\Phi}\partial\Phi - 2\mathcal{L} = -2g\psi\bar{\partial}\psi \tag{5.108}$$

Note that the stress tensor is symmetric on-shell. A modification into an identically symmetric form doesn't change the Ward Identities, so we need not modify!

$$
\begin{aligned}
T(z) &= -2\pi T_{zz} \\
&= -\pi g : \psi(z)\partial\psi(z) : \qquad (5.109)\\
&= -\pi g \lim_{w\to z} \left(\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle\right)
\end{aligned}
$$

$$(5.110)$$

Then,

$$
\begin{aligned}
T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \qquad (5.111)\\
&\sim \frac{1}{2}\frac{\partial\psi(z)}{z-w} + \frac{1}{2}\frac{\psi(z)}{(z-w)^2} \qquad (5.112)\\
&\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} \qquad (5.113)
\end{aligned}
$$

So fermion field ψ is a primary field with conformal dimension $h = \frac{1}{2}$!

And

$$
\begin{aligned}
T(z)T(w) &= \pi^2 g^2 : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \qquad (5.114)\\
&\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \qquad (5.115)
\end{aligned}
$$

We see that the stress tensor is again **not a primary field with** $h = 2$, and the OPE contains the same anomalous term as that of free Boson, but with a different numerical factor.

5.4.3 The Ghost System

$$S = \frac{1}{2}g\int d^2x b_{\mu\nu}\partial^{\mu}c^{\nu} \qquad (5.116)$$

The fields b, c are not fundamental dynamical fields. This system appears as a Jacobian in some functional integrals of string theoretic applications. Thee fields b, c are called (reparametrization) ghosts. Also, $b_{\mu\nu}$ is traceless, symmetric and anti-commuting in nature with the field c^ν.

The equations of motion can be straightforwardly seen to be -

$$\partial^\alpha b_{\alpha\mu} = 0, \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0 \tag{5.117}$$

In holomorphic coordinates, letting $b_{zz} = b, b_{\bar{z}\bar{z}} = \bar{b}$ (other components vanish in holomorphic form), the above equations take the following form -

$$\bar{\partial} b = 0, \partial \bar{b} = 0 \tag{5.118}$$

$$\bar{\partial} c = 0, \partial \bar{c} = 0, \partial c = -\bar{\partial}\bar{c} \tag{5.119}$$

We rewrite the action in a form that's ripe for the use of Gaussian Integrals as -

$$S = \frac{1}{2}\int d^2x d^2y b_{\mu\nu}(x) A^{\mu\nu}_\alpha(x, y) c^\alpha(y) \tag{5.120}$$

with $A^{\mu\nu}_\alpha(x, y) = \frac{1}{2} g \delta^\nu_\alpha \delta(x - y)\partial^\mu$. The propagator is then just given by $K = A^{-1}$ -

$$\frac{1}{2} g \delta^\mu_\alpha \partial^\nu K^\beta_{\mu\nu}(x, y) = \delta(x - y)\delta_{\alpha\beta} \tag{5.121}$$

and in holomorphic form, this is -

$$g \partial_{\bar{z}} K^\beta_{zz} = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z - w}\delta_{\beta z} \tag{5.122}$$

which gives

$$\langle b(z)c(w)\rangle = K^{z}_{zz}(z,w) = \frac{1}{\pi g}\frac{1}{z-w} \tag{5.123}$$

we thus have the following OPEs,

$$b(z)c(w) \sim \frac{1}{\pi g}\frac{1}{z-w} \tag{5.124}$$
$$c(z)b(w) \sim \frac{1}{\pi g}\frac{1}{(z-w)} \tag{5.125}$$
$$b(z)\partial c(w) \sim -\frac{1}{\pi g}\frac{1}{(z-w)^2} \tag{5.126}$$
$$\partial b(z)c(w) \sim \frac{1}{\pi g}\frac{1}{(z-w)^2} \tag{5.127}$$
$$\tag{5.128}$$

We are, as usual, interested in the stress tensor; the canonical one is

$$T^{\mu\nu}_{c} = \frac{1}{2}g\left(b^{\mu\alpha}\partial^{\nu}c_{\alpha} - \eta^{\mu\nu}b^{\alpha\beta}\partial_{\alpha}c_{\beta}\right) \tag{5.129}$$

but this needs modification since it's not symmetric. The modified tensor is

$$T^{\mu\nu}_{B} = \frac{1}{2}g\left\{b^{\mu\alpha}\partial^{\nu}c_{\alpha} + b^{\nu\alpha}\partial^{\mu}c_{\alpha} + \partial_{\alpha}b^{\mu\nu}c^{\alpha} - \eta^{\mu\nu}b^{\alpha\beta}\partial_{\alpha}c_{\beta}\right\} \tag{5.130}$$

From which we write,

$$T(z) = \pi g : (2\partial cb + c\partial b) : \tag{5.131}$$

and we then get, using Wick's theorem

$$T(z)b(w) \sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w} \tag{5.132}$$

$$T(z)c(w) \sim -\frac{-c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{(z-w)} \tag{5.133}$$

$$T(z)T(w) \sim \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{5.134}$$

We note that the fields b, c are primary with conformal dimensions $h = 2$ and $h = -1$, respectively. Once again, we see that $T(w)$ is not a primary field and contains an anomalous term with a different numerical factor!

We obtain the so-called *simple ghost system* by subtracting the total derivative $:\partial(cb):$ from $T(z)$ (and not changing any OPE between b and c).

Consider the modified theory as follows,

$$T(z) = \pi g : \partial c b : \tag{5.135}$$

The new OPEs are,

$$T(z)c(w) \sim \frac{\partial c}{z-w} \tag{5.136}$$

$$T(z)b(w) \sim \frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} \tag{5.137}$$

$$T(z)T(w) \sim \frac{-1}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{5.138}$$

We now have the conformal dimensions for b and c as $h = 1$ and $h = 0$, respectively. The numerical factor of the anomalous term in the OPE of T has changed, too.

5.5 The Central Charge and the transformation of Energy Momentum Tensor

From the above calculations, we are motivated to write the following general OPE of T with itself,

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{5.139}$$

- Here c, the *central charge* or *conformal anomaly* is a model dependent constant which can't be determined just by symmetric considerations, and requires short-distance behavior of the theory!
- Talking about the short-distance behaviour, note that we can deduce the $(z-w)^{-4}$ behaviour from the Schwinger function $\langle T(z)T(0)\rangle = \frac{\frac{c}{2}}{z^4}$ after letting $A = \frac{c}{4\pi^2}$ in equation 4.80.
- Scale invariance and Bose symmetry make $const./(z-w)^4$ the only sensible addition to OPE!
- This also measures the number of d.o.f of the system since the stress tensors of decoupled systems just add up, resulting in a central charge, which is the sum of each decoupled system. This can also be noted via the Zamolodchikov c-theorem.

So, the lesson is that $T(z)$ is not a primary field. Using the conformal Ward identity,

$$\begin{aligned} \delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint_c dz\epsilon(z)T(z)T(w) &(5.140)\\ &= -\frac{1}{12}c\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w) &(5.141)\end{aligned}$$

Exponentiating this yields

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w; z\}\right] \tag{5.142}$$

where, $\{w; z\}$ is the Schwarzian derivative given by,

$$\{w; z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2 \tag{5.143}$$

We can check its validity by using the infinitesimal form.

Properties of Schwarzian Derivative:

- *Composition Law*

$$u; z = w; z + \left(\frac{dw}{dz}\right)^2 u; w \tag{5.144}$$

-

$$\{w; z\} = -\left(\frac{dw}{dz}\right)^2 \{z; w\} \tag{5.145}$$

- $\{w : z\} = 0$ for $w(z) = \frac{az+b}{cz+d}$, $(ad - bc) = 1$. That is, for global transformation, the Schwarzian vanishes - which implies that $T(z)$ is only a Quasi-Primary field.

5.6 Physical Meaning of Central Charge

The central charge describes how a system behaves with a macroscopic length scale - introduced, for example, by 'boundary conditions' or 'curvature'. In what follows we will describe these two instances of breaking local conformal symmetry.

5.6.1 CFT on a cylinder

Consider a CFT on the whole complex plane, with coordinates $z, \bar{z}$. We can map this theory to a cylinder by the following transformation,

$$z \longrightarrow w = \frac{L}{2\pi} \ln z \tag{5.146}$$

Under such a transformation,

$$\frac{dw}{dz} = \frac{L}{2\pi z} \implies \{w; z\} = \frac{1}{2z^2} \tag{5.147}$$

So according to equation 5.142,

$$T_{cyl.}(w) = \left(\frac{2\pi}{L}\right)^2 \left[T_{pl.}z^2 - \frac{c}{24}\right]. \tag{5.148}$$

If the vacuum energy density of the theory vanishes on the plane then we have $\langle T_{cyl.}(w)\rangle = -\frac{c\pi^2}{6L^2}$ meaning, the central charge is proportional to the change in vacuum energy density due to the periodic boundary conditions of the cylinder (the Casimir Energy) - given by the scale L. Note that this energy goes naturally to zero as we remove the macroscopic scale ($L \to \infty$).

5.6.2 CFT on a curved 2d manifold

When a conformal field theory is done on an arbitrary 2d manifold, there is a macroscopic scale because of the Scalar curvature R. This makes the trace of the energy-momentum tensor non-zero and proportional to the central charge

$$\langle T^{\mu}_{\mu}(x)\rangle_g = \frac{c}{24\pi}R(x). \tag{5.149}$$

Again, this vanishes when we remove the scale in the theory ($R \to 0$) corresponding to the flat space, where we have indeed derived this to be zero.

Chapter 6

Operator Formalism of 2d CFT

In this Chapter, we will study the quantum structure of conformal field theory from a canonical perspective. The path integral approach is helpful in general while considering the behaviour of correlation function under symmetries, but the operator formalism allows us to use many algebraic techniques to learn about the QFT. Since we are dealing with Euclidean spacetime, we will first start by choosing a time direction in our 2d space and defining the inner product on the Hilbert space. We then define mode expansions and radial ordering and relate OPEs to commutators. We will then derive the Virasoro Algebra between the quantum generators of conformal transformations (on the Hilbert space). We then look at how primary fields help us construct the descendant states from the vacuum (or the asymptotic states), which are Hamiltonian eigenstates and closed under the Virasoro generators, forming a Module under Virasoro algebra (called Verma Module). We will explicitly realise this structure by considering free massless bosons with various boundary conditions. We will then go deeper into the structure of CFTs, by considering normal ordering for fields which are not *free*, then obtaining the so-called *descendant fields* that generate the descendant states when applied on vacuum. This leads to conformal families and the OPEs of descendants with stress tensor. This makes the *Operator Algebra* more natural, including all the OPE's regular terms. Two-point correlators are then considered, and using the symmetry properties; the operator product coefficients are computed up to three-point correlator coefficients for which we need further dynamical input. The process of finding any n-point correlation function is indicated and explicitly shown for the 4-point function, where we express the form in terms of *conformal blocks* (which can be computed just by symmetries) and the three-point correlators. Further analysis of the 4-point function naturally leads to crossing symmetry (which is the dynamical input, so to speak), which might constrain the three-point coefficients - called the Conformal Bootstrap way of solving a particular CFT completely.

6.1 Radial Quantization

In 2 Dimensions, the time direction can be chosen naturally to be the radial direction on the complex plane, thus the name.

6.1.1 Map to Cylinder - Choose the time direction

By *naturally* - we mean the following:

We can compactify the space part of $\mathbb{R}^2$ as follows

$$t \in (-\infty, \infty) \tag{6.1}$$

$$x \in [0, L);\ x + 2\pi L \equiv x \tag{6.2}$$

And in complex coordinates, this takes the form

$$\xi = t + ix,\ \bar{\xi} = t - ix. \tag{6.3}$$

We map these coordinates onto a Riemann Sphere via,

$$z = \exp\left(2\pi\frac{\xi}{L}\right),\ \bar{z} = \exp\left(2\pi\frac{\bar{\xi}}{L}\right) \tag{6.4}$$

and we note that,

$$t \to -\infty \equiv |z| \to 0 \implies z = 0, \tag{6.5}$$

$$t \to \infty \equiv |z| \to \infty \implies z \to \infty \tag{6.6}$$

Thus, whenever we see 0 or ∞ popping in the fields' arguments, we will talk about something *asymptotic*. So we essentially see that $|z|$, the radial coordinate on the full complex plane (mapping from the cylinder), represents the time direction.

Like in any QFT, we will assume the existence of a vacuum state upon which the Hilbert space will be constructed (via creation operators). And in any interacting field theory, we hold the asymptotic fields $\phi_{in} \propto \lim_{t\to-\infty} \phi(x,t)$ and $\phi_{out} = \lim_{t\to\infty} \phi(x,t)$ to be free.

And we define the asymptotic in-state as follows,

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle \tag{6.7}$$

6.1.2 Inner Product

Hermitian conjugate in Minkowski space doesn't affect the space-time coordinates, but when going to the Euclidean version, i.e. setting $\tau = it$, we must reverse the time coordinate under a Hermitian conjugation. Which translates to $z \to \frac{1}{z^*}$ in our coordinates. Thus, the following Hermitian conjugation of the fields makes sense (on the real surface $\bar{z} = z^*$),

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \tag{6.8}$$

where ϕ is a quasi-primary with conformal dimensions $(h, \bar{h})$. Using these pre-factors has the following advantages,

$$
\begin{aligned}
\langle\phi_{out}|\phi_{in}\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\phi(z,\bar{z})^\dagger \phi(w,\bar{w})|0\rangle \\
&= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{2\bar{h}} \langle 0|\phi(\frac{1}{\bar{z}},\frac{1}{z})\phi(0,0)|0\rangle \\
&= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0|\phi(\bar{\xi},\xi)\phi(0,0)|0\rangle \\
&= C
\end{aligned}
\tag{6.9}
$$

where we have let $\langle\phi_{out}| = |\phi_{in}\rangle^\dagger$ and used the form of two point correlator as $\langle\phi(\bar{\xi},\xi)\phi(0,0)\rangle = \frac{C}{\bar{\xi}^{2h}\xi^{2\bar{h}}}$ (already time-ordered). So $\langle\phi_{out}|\phi_{in}\rangle$ is just a constant factor independent of ξ and thus well defined!

6.1.3 Mode Expansions

We expand the conformal field $\phi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ into modes as,

$$
\phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{6.10}
$$

This is just a Laurent series that respects conformal scaling. The modes are then given by

$$
\begin{aligned}
\frac{1}{2\pi i}\oint \phi(z,\bar{z})\bar{z}^{n+\bar{h}-1}dz &= \frac{1}{2\pi i}\sum_{\tilde{m},\tilde{n}\in\mathbb{Z}} z^{-\tilde{m}-h}\oint \frac{\bar{z}^{-1}}{\bar{z}^{n-\tilde{n}}}\phi_{\tilde{m}\tilde{n}} \\
&= \sum_{\tilde{m}\in\mathbb{Z}} z^{-\tilde{m}-h}\phi_{\tilde{m}n}
\end{aligned}
$$

Perform one more integral over $z^{m+h-1}d\bar{z}$ to write,

$$
\phi_{mn} = \frac{1}{2\pi i}\oint z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}) \tag{6.11}
$$

From 6.11 and 6.8 we can get the usual expression,

$$
\phi^\dagger_{mn} = \phi_{-m,-n} \tag{6.12}
$$

which also justifies the $h, \bar{h}$ in the mode expansions.

The annihilation of the vacuum by the positive field modes can be obtained from the well- behavedness of $|\phi_{in}\rangle$, $|\phi_{out}\rangle$:

$$\begin{aligned} |\phi_{in}\rangle &= \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle \\ &= \lim_{z,\bar{z}\to 0} \sum_{m,n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{mn}\,|0\rangle \end{aligned}$$

Thus, in order to deal with the singular terms in the asymptotic limit; we let

$$\phi_{mn}\,|0\rangle = 0 \quad \left(m > -h, n > -\bar{h}\right) \tag{6.13}$$

We can lighten up the notation by hiding the anti-holomorphic dependence of the field, keeping in mind that while there is $\bar{z}$ dependence - in most of the cases, there is decoupling between hol. and anti-hol. parts in 2d CFT.

$$\phi(z) \sim \sum_{m\in\mathbb{Z}} z^{-m-h}\phi_m \tag{6.14}$$

$$\phi_m = \frac{1}{2\pi i}\oint dz z^{m+h-1}\phi(z) \tag{6.15}$$

6.1.4 Radial Ordering and OPE

$$\mathcal{T}\phi_1(z)\phi_2(w) \equiv \mathcal{R}\phi_1(z)\phi_2(w) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \\ \phi_2(w)\phi_1(z) & |z| < |w| \end{cases} \tag{6.16}$$

The l.h.s of an OPE must be radially ordered!

Consider the following integral, which can be evaluated via the corresponding OPE

$$\begin{aligned} \oint_w dz a(z)b(w) &= \int_{C_1} dz a(z)b(w) - \int_{c_2} dz b(w)a(z) \\ &= [A, b(w)]\,;\; A = \oint a(z)dz \end{aligned} \tag{6.17}$$

The contour C_1 encircles $0, w$ with a $|z| = |w| + \epsilon$ and C_2 encircles only the 0 with $|z| = |w| - \epsilon$. We then take the limit $\epsilon \to 0$. Thus, the above is essentially an equal-time commutator!

We thus write,

$$[A, B] = \oint dw \oint_w dz b(w) a(z) \tag{6.18}$$

where $A = \oint a(z)dz$ and $B = \oint b(z)dz$. Unless explicitly mentioned otherwise, the contour integrals are around the point $z = 0$.

6.2 Virasoro Algebra

Recall the conformal ward Identity,

$$\delta_{\epsilon\bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \tag{6.19}$$

We can then define the conformal charge to be

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \tag{6.20}$$

and we can see that the ward identity gives $\delta_\epsilon \phi(w) = -\left[Q_\epsilon, \phi(w)\right]$! So Q_ϵ generates the conformal transformations on the Hilbert space.

We then expand the stress tensor into modes L_n,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n,\ L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \tag{6.21}$$

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1} \epsilon_n \tag{6.22}$$

similarly for $\bar{T}(z), \bar{\epsilon}(z)$. Note that $Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n$. So, the mode operators $L_n, \bar{L}_n$ are the generators of the local conformal transformation on the Hilbert space. We had similar generators (forming the Witt Algebra) on the space of functions.

We can derive the algebra between these generators using the OPE between the stress tensors according to 6.18 and 5.139 to be as follows,

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n \left(n^2 - 1\right) \delta_{n+m,0} \\
\left[L_n, \bar{L}_m\right] &= 0 \\
\left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\bar{L}_{n+m} + \frac{c}{12} n \left(n^2 - 1\right) \delta_{n+m,0}
\end{aligned} \tag{6.23}
$$

This is called the Virasoro Algebra. We note the additional central charge terms from the Witt Algebra. But note that,

$$
[L_{\mp 1}, L_0] = \mp L_{\mp 1} \qquad [L_1, L_{-1}] = 2L_0 \tag{6.24}
$$

which is the same algebra obtained for Witt generators $l_{-1,0,1}$.

We can conclude that generators of $SL(2,\mathbb{C})$ transformations (global conformal transformations) on the Hilbert Space are again given by L_{-1}, L_0, L_1. Here also, we see that

- $L_0 + \bar{L}_0$ generate the dilatations $(z, \bar{z}) \to \lambda(z, \bar{z})$ which correspond to the time translations. So the Hamiltonian of any CFT in 2D must be proportional to $L_0 + \bar{L}_0$!
- $i(L_0 - \bar{L}_0)$ generate rotations.
- $L_{-1}, \bar{L}_{-1}$ generate translations.
- $L_1, \bar{L}_1$ generate special conformal transformations.

We deduce the above by noting the relevant non-zero terms in $\epsilon(z), \bar{\epsilon}(\bar{z})$.

6.3 Hilbert Space

We can deduce a lot of information about the Hilbert Space of CFT just by the Virasoro Generators and the primary fields.

1. The conformal invariance of the vacuum $\implies$ $L_{-1,0,1}$ annihilates the vacuum state $|0\rangle$. Since the Hamiltonian is given by L_0, this fixes the ground state energy to be zero! Another way to achieve this is by imposing the well-behavedness of $T(z)|0\rangle, \bar{T}(\bar{z})|0\rangle$ as $z, \bar{z} \to 0$. This gives us

$$L_n |0\rangle = 0, \ \bar{L}_n |0\rangle = 0. \ (n \geq -1)\,. \tag{6.25}$$

2. Primary Fields acting on the vacuum create asymptotic states, also the Hamiltonian's eigen states. Using the OPE between $T(z)$ and a primary field $\phi(w, \bar{w})$ we have the following,

$$\left[L_n, \phi(w, \bar{w})\right] = h(n+1)w^n \phi(w, \bar{w}) + w^{n+1} \partial\phi(w, \bar{w}) \quad (n \geq -1) \tag{6.26}$$

similarly with $\bar{L}_n$.

We define $|h,\bar{}\rangle = \phi(0,0)|0\rangle$ as the primary state or the highest weight state with the following behaviour implied by equation 6.26,

$$L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle \tag{6.27}$$

$$\bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle \tag{6.28}$$

$$L_n |h, \bar{h}\rangle = 0, \ \bar{L}_n |h, \bar{h}\rangle = 0 \quad (n > 0) \tag{6.29}$$

3. Excited states of the states $|h, \bar{h}\rangle$ can be obtained by the raising operators $\phi_{-m}, L_{-m}\ (m > 0)$ which raise the conformal dimension h by m - where ϕ_m are the modes of the primary field $\phi(w, \bar{w})$ defined in 6.15.

$$\left[L_n, \phi(m\right] = (n(h-1) - m)\, \phi_{n+m} \tag{6.30}$$

$$\left[L_0, \phi_m\right] = -m\phi_m \tag{6.31}$$

Thus, an excited state called the descendant state may be obtained as

$$L_{-k_1} \dots L_{-k_n} |h\rangle\,, \tag{6.32}$$

with $1 \leq k_1 \leq \dots \leq k_n$ as a convention for the ordering. The conformal dimension of the descendant state is $h' = h + k_1 + \dots + k_n \equiv h + N$, and we call N the level of the descendant. The number of linearly independent N level descendants are just given the partitions of the integer N, $p(N)$.

4. These descendant states are closed under the conformal transformations given by L_n. The space of an asymptotic state $|h\rangle$ and its descendants forms a representation/module under the Virasoro algebra. This is the Verma Module.

6.4 Free Boson

Now let's see all this manifest in the free massless Boson on a cylinder with a circumference of L. We start with the following Lagrangian on 1+1 Minkowski Space,

$$\mathcal{L} = \frac{1}{2} g \int dx \left((\partial_t \varphi)^2 - \left(\partial_x \varphi\right)^2 \right) \tag{6.33}$$

and with boundary conditions $\varphi\ (x + L, t) \equiv \varphi(x, t)$. Since the field is periodic in space with a period L, we can expand it readily using the discrete Fourier series,

$$\varphi(x,t) = \sum_{n\in\mathbb{Z}} e^{2\pi inx/L}\varphi_n(t) \tag{6.34}$$

$$\varphi_n(t) = \frac{1}{L}\int dxe^{-2\pi inx/L}\varphi(x,t) \quad (\varphi_n^\dagger = \varphi_{-n}) \tag{6.35}$$

One can then rewrite the Lagrangian in terms of these modes to find out the conjugate momenta and the Hamiltonian. We promote the fields into operators and impose the usual commutation relation, $\left[\varphi_n, \pi_m\right] = i\delta_{mn}$

$$\mathcal{L} = \frac{1}{2}gL\sum_n \left\{\dot{\varphi}_n\dot{\varphi}_{-n} - \left(\frac{2\pi n}{L}\right)^2 \varphi_n\varphi_{-n}\right\} \tag{6.36}$$

$$\pi_n = gL\dot{\varphi}_{-n} \quad (\pi_n^\dagger = \pi_{-n}) \tag{6.37}$$

$$H = \frac{1}{2gL}\sum_n \left\{\pi_n\pi_n^\dagger + (2\pi ng)^2\varphi_n\varphi_{-n}\right\} \tag{6.38}$$

The above Hamiltonian describes a decoupled oscillator system with frequencies $\omega_n = 2\pi\left|n\right|/L$.

Since here there is a zero frequency mode ϕ_0, the usual definition of ladder operators via

$$\tilde{a}_n = \frac{1}{\sqrt{2\omega_n}}\left(\omega_n\phi_n + i\pi^\dagger\right) \tag{6.39}$$

doesn't work. We instead define,

$$a_n = \begin{cases} -i\sqrt{n}\tilde{a}_n & (n>0) \\ i\sqrt{-n}\tilde{a}_{-n}^\dagger & (n<0) \end{cases} \tag{6.40}$$

$$\bar{a} = \begin{cases} -i\sqrt{n}\tilde{a}_{-n} & (n>0) \\ i\sqrt{-n}\tilde{a}_n^\dagger & (n<0) \end{cases} \tag{6.41}$$

and treat the zero mode φ_0 separately. The commutation relations then differ from the usual commutation rules obeyed by $\tilde{a}_n$,

$$[a_n, a_m] = n\delta_{n+m} \qquad [a_n, \bar{a}_m] = 0 \qquad [\bar{a}_n, \bar{a}_m]\, n\delta_{n+m} \tag{6.42}$$

We can then write the Hamiltonian as,

$$H = \frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L}\sum_{n>0}\left(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n\right) \tag{6.43}$$

with

$$[H, a_{-m}] = \frac{2\pi}{L}ma_{-m}, \quad [H, \bar{a}_{-m}] = \frac{2\pi}{L}m\bar{a}_{-m} \tag{6.44}$$

So, a_{-m} $(m > 0)$ raises the energy of an eigenstate of H by $\frac{2m\pi}{L}$.

We can then express the field φ at $t = 0$ as

$$\varphi(x) = \varphi_0 + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n - \bar{a}_{-n}\right)e^{2\pi inx/L} \tag{6.45}$$

One can obtain the Heisenberg operators (time evolution) of these fields via the Hamiltonian 6.43 to be,

$$\begin{aligned}\varphi_0(t) &= \phi_0(0) + \frac{1}{gL}\pi_0 t\\ a_n(t) &= a_n(0)e^{-2\pi int/L}, \bar{a}_n(t) = \bar{a}_n(0)e^{-2\pi int/L}\end{aligned} \tag{6.46}$$

We thus find the full mode expansion of the field at any time t to be,

$$\varphi(x,t) = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi in(x-t)/L} - \bar{a}_{-n}e^{2\pi in(x+t)/L}\right) \tag{6.47}$$

In order to realize the structure imposed by the 2D conformal symmetry, we must go to the complex plane. We do so by first going to Euclidean space-time by taking $t \to -i\tau$ and then mapping the cylinder to the complex plane via the familiar transformation,

$$z = e^{2\pi(\tau - ix)/L} \quad \bar{z} = e^{2\pi(\tau + ix)/L}. \tag{6.48}$$

In these complex coordinates, the mode expansion of φ reads,

$$\varphi(z,\bar{z}) = \varphi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n}\left(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}\right) \tag{6.49}$$

We immediately notice that this is far from the mode expansion of a primary field (6.14),

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n. \tag{6.50}$$

We already know that φ is not a primary field, but its derivative is. We find that,

$$i\partial\varphi(z) = \frac{1}{4\pi g}\frac{\pi_0}{z} + \frac{1}{\sqrt{4\pi g}} \sum_{n\neq 0} a_n z^{-n-1} \tag{6.51}$$

And thus we are finally motivated to define a_0 using the zero mode π_0,

$$a_0 \equiv \bar{a}_0 = \frac{\pi_0}{\sqrt{4\pi g}} \tag{6.52}$$

which puts $i\partial\varphi$ indeed in the form of 6.14!

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi g}} \sum_{n} a_n z^{-n-1} \tag{6.53}$$

We note the conformal dimension $h = 1$. The case of $\bar{\partial}\varphi(\bar{z})$ is similar with $\bar{h} = 1$.

Although the field φ is not primary, it allows us to construct other primary fields - called vertex operators:

$$\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\varphi(z,\bar{z})}: \tag{6.54}$$

In terms of the field modes, the normal ordering splits the exponential into two exponentials, one on the left containing creation operators and another with annihilation operators placed to the right.

Recall from 5.89 that $T(z) = -2\pi g : \partial\varphi(z)\partial\varphi(z) :$ for the massless free Boson.

We can then compute the OPE of $\mathcal{V}_\alpha$ with $T(z)$ by writing the exponential as a sum and using Wick's theorem,

$$\begin{aligned} T(z)\mathcal{V}_\alpha(w,\bar{w}) =& -2\pi g \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} : \partial\varphi(z)\partial\varphi(z) :: \varphi(w,\bar{w})^n : \\ &\sim -\frac{1}{8\pi g}\frac{1}{(z-w)^2}\sum_{n=2}^{\infty}\frac{(i\alpha)^n}{(n-2)!} : \varphi(w,\bar{w})^{n-2} : \\ &+ \frac{1}{z-w}\sum_{n=1}^{\infty}\frac{(i\alpha)^n}{n!} n : \partial\varphi(z)\varphi(w,\bar{w})^{n-1} : \\ &\sim \frac{\alpha^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{z-w} \end{aligned} \tag{6.55}$$

We thus note that $\mathcal{V}_\alpha$ is a primary field with conformal dimension $h = \frac{\alpha^2}{8\pi g}$. It also follows that $\bar{h} = \frac{\alpha^1}{8\pi g}$.

The OPE between these vertex operators can be computed by using the following relation,

$$: e^{A_1} :: e^{A_2} :=: e^{A_1+A_2} : e^{\langle A_1 A_2\rangle} \tag{6.56}$$

where A_1, A_2 are some linear combination of annihilation and creation operators. Applied this to field $\mathcal{V}_\alpha$ and φ gives,

$$\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta/4\pi g}\mathcal{V}_{\alpha+\beta}(w,\bar{w}) + \cdots \tag{6.57}$$

But Conformal Invariance fixes the form of such two-point correlation functions, i.e.

$h_\alpha = h_\beta \implies \alpha^2 = \beta^2$ and we also impose $\alpha\beta < 0$ so that the functions are local and decay with distance - so the above correlator doesn't vanish for $\alpha = -\beta$,

$$\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_{-\alpha}(w,\bar{w}) \sim |z-w|^{-2\alpha^2} + \cdots \tag{6.58}$$

The general result is that the n-point correlators of vertex operators vanish unless $\sum_{i=1}^{n} \alpha_i = 0$.

We will now construct the Fock Space, where we will also see the role of Vertex Operators.

1. We first note that $a_n, \bar{a}_n$ are annihilation operators for $n > 0$ and creation operators for $n < 0$. And since π_0 occurring in the Hamiltonian commutes with all of the other terms $(a_n, \bar{a}_n)$, we can label the eigenstates of Hamiltonian with the eigenvalues of π_0. We see that the vacuum is now a one-parameter family of vacua $|\alpha\rangle$ where α is the eigenvalue of $a_0 = \pi_0/\sqrt{4\pi g}$. That is,

$$a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle = 0 \quad (n > 0) \quad \text{where} \quad a_0|\alpha\rangle = \bar{a}_0|\alpha\rangle = \alpha|\alpha\rangle \tag{6.59}$$

2. The energy-momentum tensor (holomorphic) is given by

$$\begin{aligned} T(z) &= -2\pi g : \partial\varphi(z)\partial\varphi(z) : \\ &= \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : a_n a_m : \end{aligned} \tag{6.60}$$

We immediately compare this with the mode expansion 6.21 and write,

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m \quad (n \neq 0) \\ L_0 &= \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 \end{aligned} \tag{6.61}$$

And we recover the expected Hamiltonian form for a 2D CFT,

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0\right). \tag{6.62}$$

3. We see that a_m and L_m behave similarly to the Hamiltonian (or L_0). We construct the excited stated by successively acting a_{-m} and $\bar{a}_{-n}$ on the vacua $|\alpha\rangle$.

$$a_{-1}^{n_1} a_{-2}^{n_2} \cdots a_{-1}^{-m_1} a_{-2}^{-m_2} \cdots |\alpha\rangle \qquad (n_i, m_j \geq 0) \tag{6.63}$$

Note that $|\alpha\rangle$ has a conformal dimension of $\frac{\alpha^2}{2}$ given by the action of L_0! So, these states are the eigenstates of L_0, $\bar{L}_0$ with conformal dimensions (eigenvalues)

$$h = \frac{1}{2}\alpha^2 + \sum_j jn_j, \quad \bar{h} = \frac{1}{2}\alpha^2 + \sum_j jm_j \tag{6.64}$$

This is, of course, reminiscent of the descendant states 6.32.

4. The punchline of this entire construction is given by the following claim,

$$|\alpha\rangle = \mathcal{V}_\alpha(0)\,|0\rangle \tag{6.65}$$

The vertex operator, being a primary field, essentially creates an asymptotic eigenstate of the Hamiltonian (the primary state or the highest weight state) from which we can build the descendant states.

Using the mode expansions, we can now compute the two-point functions. We are keener about the correlator of the primary field $\partial\varphi$,

$$\langle \varphi(z)\partial\varphi(w)\rangle = \sum_{m,n\neq=} \frac{1}{n}\langle a_n, a_m\rangle\, z^{-n} w^{-m-1} \tag{6.66}$$

$$= \frac{1}{w}\sum_{n>0}\left(\frac{w}{z}\right)^n \tag{6.67}$$

$$= \frac{1}{z-w} \tag{6.68}$$

where we have used,

$$\langle a_n a_m \rangle = \begin{cases} \langle a_m a_n + [a_n, a_m] \rangle = n\delta_{n+m} & n > 0 \\ 0 & n \le m \end{cases} \tag{6.69}$$

Note that we have used the holomorphic part of $\varphi(z, \bar{z})$ since it's the only term that contributes to the $\langle \partial\varphi \rangle\, \partial\varphi$.

Differentiating equation 6.68 with respect to z gives,

$$\langle \partial\varphi(z)\partial\varphi(w) \rangle = -\frac{1}{(z-w)^2} \tag{6.70}$$

which is the same as the one we derived using the Path Integral formulation. We can compute the vacuum energy density using,

$$\begin{aligned} T(z) &= -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : \\ \langle T(z) \rangle &= -\frac{1}{2} \lim_{\epsilon \to 0} \left(\langle \partial\varphi(z+\epsilon)\partial\varphi(z) \rangle + \frac{1}{\epsilon^2} \right) \end{aligned} \tag{6.71}$$

which is just zero in the light of 6.70. And when we go back to the cylinder, we know from our central charge discussion that we must modify this vacuum density as,

$$\langle T(z) \rangle_{cyl.} = -\frac{1}{24} \left(\frac{2\pi}{L} \right)^2 \tag{6.72}$$

where we have used $c = 1$. This allows us to fix the constants in L_0,

$$L_0 = \begin{cases} \sum_{n>0} a_{-n} a_n & \text{Plane} \\ \\ \sum_{n>0} a_{-n} a_n - \frac{1}{24} & \text{Cylinder} \end{cases} \tag{6.73}$$

Hamiltonian on the cylinder is then given by,

$$H = \frac{2\pi}{L}\left((L_0)_{cyl.} + \left(\bar{L}_0\right)_{cyl.}\right) \tag{6.74}$$

$$= \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{1}{12}\right) \tag{6.75}$$

More generally, we note that,

$$H_{cyl.} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \tag{6.76}$$

where L_0 are the modes of stress tensor on the plane.

6.4.1 Twisted Boundary conditions

Since, the Lagrangian is quadratic we can also choose *anti-periodic* (or *twisted*) boundary conditions for the field φ, i.e. $\varphi(x+L,t) \equiv -\varphi(x,t)$ This has the following effect on the theory:

1. From the Fourier series, we can immediately conclude that $\varphi_0 = 0$ and the summation of modes is over half integers n.

2. The ladder operators are defined the same way, obeying the same commutation relations as before but labeled by half-integers.

3. The field now is a multi-valued function on the cylinder. One can define the theory on two Riemann Sheets when mapped to the complex plane.

4. If we define the operator G that takes φ to $-\varphi$, via $G\varphi G^{-1} = -\varphi$ (i.e. G allows us to move between the Riemann Sheets), then it follows that

$$G\varphi G^{-1} = \varphi \implies \left\{G, \varphi\right\} = 0 \implies \{G, a_n\} = 0; \quad [G, H] = 0 \tag{6.77}$$

And $G^2 = 1$ implies that every eigenstate of H is degenerate corresponding to ± 1 eigenvalues of G. So, unlike the periodic case where we had vacua for each α (corresponding to $\mathcal{V}_\alpha$), we have a doubly degenerate vacuum - $|0_+\rangle$, $|0_-\rangle$ (corresponding to G).

5. Going back to equation 6.67, we can rewrite this sum over half-integers instead for the anti-periodic case to get

$$\langle \varphi(z)\partial\varphi(w)\rangle = \frac{1}{w}\sqrt{\frac{w}{z}\frac{1}{z-w}} \tag{6.78}$$

which, upon differentiating with z gives,

$$\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{2}\frac{\sqrt{z/w}+\sqrt{w/z}}{(z-w)^2} \tag{6.79}$$

6. We can immediately calculate the central charge,

$$\begin{aligned}
T(z)T(w) &= \frac{1}{4} : \partial\varphi(z)\partial\varphi(z) :: \partial\varphi(w)\partial\varphi(w) : \\
&\sim -\frac{1}{4}\left\{\frac{1}{4}\frac{\left(\sqrt{z/w}+\sqrt{w/z}\right)^2}{(z-w)^4} - 4\frac{1}{2}\frac{\left(\sqrt{z/w}+\sqrt{w/z}\right) : \partial\varphi(z)\partial\varphi(w) :}{(z-w)^2}\right\} \\
&\sim \frac{-1/4}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
\end{aligned} \tag{6.80}$$

So, $c = -1/2$ for the anti-periodic boundary conditions. Note that we took the limit $z \longrightarrow w$, which gives the complete form of the singularities.

7. With the two-point function 6.79 the vacuum energy density on the plane(according to 6.71) becomes,

$$\langle T(z)\rangle = -\frac{1}{2}\lim_{\varepsilon\to 0}() \tag{6.81}$$

8. On the cylinder, due to the central charge we,

$$\langle T(z)\rangle_{cyl.} = \frac{1}{48}\left(\frac{2\pi}{L}\right)^2 \tag{6.82}$$

9.

$$L_0 = \begin{cases} \sum_{n>0} a_{-n} a_n + \frac{1}{16} & \text{Plane} \\ \\ \sum_{n>0} a_{-n} a_n + \frac{1}{16} - \frac{1}{48} & \text{Cylinder} \end{cases} \tag{6.83}$$

Hamiltonian on the cylinder is then given by,

$$\begin{aligned} H &= \frac{2\pi}{L}\left((L_0)_{cyl.} + (\bar{L}_0)_{cyl.}\right) \\ &= \frac{2\pi}{L}\left(L_0 + \bar{L}_0 + \frac{1}{24}\right) \end{aligned} \tag{6.84}$$

6.5 Conformal Families

We have adopted the following procedure of subtracting the vacuum expectation value from the product of fields in order to normally order it,

$$: AB := (AB - \langle AB \rangle) \tag{6.85}$$

but considering fields like $T(z)$, subtracting the expectation value only removes the lower order singularities, and there is a remaining singularity due to the central charge,

$$: T(z)T(z) :\sim \lim_{z \to w} \frac{c/2}{(z-w)^4} \tag{6.86}$$

In this sense, fields like $T(z)$ are not *free*. We generalize the procedure of normal ordering to the product of such fields by subtracting all the singularities from the product expansion, i.e.

If the OPE of A, B is written as,

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{6.87}$$

then, we define the generalized normal ordering as

$$(AB)(w) = \{AB\}_0(w) \tag{6.88}$$

$$= \lim_{z \to w} [A(z)B(w) - CA(z)B(w)] \tag{6.89}$$

$$\equiv \frac{1}{2\pi i} \oint_w \frac{dz}{z-w} A(z)B(w) \tag{6.90}$$

where the contraction $CA(z)B(w)$ now includes all the singular terms of OPE,

$$CA(z)B(w) = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{6.91}$$

With this formulation, we note that the OPE can be written as,

$$A(z)B(w) = CA(z)B(w) + (A(z)B(w)) \tag{6.92}$$

where the regular part is given by,

$$(A(z)B(w)) = \sum_{K \geq 0} \frac{(z-w)^k}{k!} \left(\partial^k AB\right)(w) \tag{6.93}$$

Representing normal ordering in terms of the contour integral (6.90) is very convenient.

We can apply some of these results on the OPE of $T(z)$ and some arbitrary field $A(w)$ to gain some insights,

First, expand $T(z)$ around w as,

$$T(z) = \sum_{n \in \mathbb{Z}} (z-w)^{-n-2} L_n(w) \tag{6.94}$$

$$L_n(w) = \frac{1}{2\pi i} \oint_w dz (z-w)^{n+1} T(z) \tag{6.95}$$

then,

$$T(z)A(w) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-2} \left(L_n A\right)(w) \tag{6.96}$$

$\left(L_n A\right)(w)$ defines the regular composite fields in the OPE (6.87), which can be deduced by comparing with the Taylor expansion 6.91 and the singular terms of OPE for a primary field,

$$T(z)A(w) = \cdots + \frac{h_A A(w)}{(z-w)^2} + \frac{\partial A(w)}{(z-w)} + (TA)(w) + (z-w)(\partial TA)(w) + \cdots \tag{6.97}$$

We thus note,

$$\left(L_0 A\right)(w) = h_A A(w) \tag{6.98}$$

$$\left(L_{-1} A\right)(w) = \partial A(w) \tag{6.99}$$

$$\left(L_{-n-2} A\right)(w) = \frac{1}{n!}\left(\partial^n TA\right)(w) \tag{6.100}$$

L_0, L_{-1} clearly reflects the nature of scaling and translation generators. For primary fields, $\left(L_n \phi\right) = 0$ for $n > 0$.

Using the contour integral version of the normal ordering and the mode expansions of fields around arbitrary points, we can derive the following normal ordering between the modes. If we write

$(AB)(z) = \sum z^{-n-h_A-h_B}(AB)_n$, we get:

$$(AB)_m = \sum_{n\le -h_A} A_n B_{m-n} + \sum_{n> -h_A} B_{m-n} A_n \tag{6.101}$$

This expression is very useful because it allows us to compute the form of the composite fields explicitly in terms of the modes of individual fields! Few of the composite fields are familiar in the case of OPEs with $T(z)$, and the above is the generalization to arbitrary OPEs.

With the above machinery in mind, we can now understand how, for every descendant state, there is a corresponding field descendant to the associated primary field. We already know that the primary fields manufacture the descendant states, leading to an infinite number of (descendant) states. We see the correspondence as follows:

Consider a descendant $L_{-n}|h\rangle$ due to a primary field ϕ, then,

$$L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle = \frac{1}{2\pi i} = \oint dz z^{1-n} T(z)\phi(0)|0\rangle \tag{6.102}$$

$$\equiv \left(L_{-n}\phi\right)(0)|0\rangle \tag{6.103}$$

the last line follows from the definition 6.96

We then extend the above and define the descendant field associated with the state $L_{-n}|h\rangle$ to be,

$$\phi^{-n}(w) \equiv \left(L_{-n}\phi\right)(w) = \frac{1}{2\pi i}\oint_w dz \frac{T(z)\phi(w)}{(z-w)^{n-1}} \tag{6.104}$$

We connect the machinery of the normal ordering and OPE to note that the above fields are just the fields that appear in the OPE of $T_(z)$ with $\phi(w)$. So from 6.98

$$\phi^{(0)}(w) = h\phi(w) \quad \phi^{-1}(w) = \partial\phi(w) \tag{6.105}$$

The importance of primary fields in a CFT is clear: these define the asymptotic states on which we can build Hamiltonian's higher conformal dimensional eigenstates. We now see that these primary fields also have descendant fields. In particular note that, $T(w)$ is a level 2 descendant of the identity operator $(L_{-2}\mathbb{I})$! And $\partial\phi$ is a level 1 descendant of the primary field ϕ. The set containing the primary field and its descendants is called a conformal family (denoted $[\phi]$).

A correlator of form $\langle\left(L_{-n}\phi\right)(w)X\rangle$ where $X = \phi_1(w_1)\ldots\phi_N(w_N)$ is a collection of primary fields, can be computed using the OPE of $T(z)$ with primaries:

$$\langle \phi^{-n}(w) X \rangle = \mathcal{L}_{-n} \langle \phi(w) X \rangle \quad (n \geq 1) \tag{6.106}$$

where,

$$\mathcal{L}_{-n} \equiv \sum_i \left\{ \frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}} \partial_{w_i} \right\} \tag{6.107}$$

Similar result follows for higher order descendants corresponding to $L_{-k} L_{-n} |0\rangle$,

$$\phi^{(-k,-n)}(w) = (L_{-k} L_{-n} \phi)(w) \tag{6.108}$$

$$= \frac{1}{2\pi i} \oint_w dz (z-w)^{1-k} T(z) (L_{-n}\phi)(w) \tag{6.109}$$

$$\langle \phi^{(-k_1,\ldots,-k_n)(w)} X \rangle = \mathcal{L}_{-k_1} \ldots \mathcal{L}_{-\|\backslash} \langle \phi(w) X \rangle \tag{6.110}$$

Since L_0, L_{-1} are the generators for dilations and translations, it follows that

$$\phi^{(0,-n)}(w) = (h+n)\, \phi^{(-n)}(w) \quad \phi^{(-1,-n)}(w) = \partial_w \phi^{(-n)}(w). \tag{6.111}$$

The general result and conclusion of the above is that any correlators involving a collection of descendant fields and primary fields can all be converted to correlators of primary fields, whose 2-point and 3-point forms are already restricted from the symmetries of CFT. It follows from the definition that the conformal family is closed under conformal transformations, which are essentially generated by L_n. This translates to the statement that the OPE between $T(z)$ a member of the conformal family involves just the other members in it. In fact, this can be made explicit by using the already known singularities in the OPE of $T(z)$ with itself in computing $T(z)\phi^{(-n)}(w)$. It is thus fruitful to organize a 2D CFT into conformal families, which are closed under the OPEs with the stress tensor, and any correlator involving the descendants can be calculated using the correlators of primaries.

6.6 Operator Algebra and Correlation functions

1. From above sections, it is clear that the problem of finding correlation functions in a CFT boils down to the correlation functions of primary field. Symmetries allowed us to restrict the form of the 2-point and 3-point correlators of such fields up to some coefficients. For the 2-point correlators, we had:

$$\langle \phi_i(w, \bar{w}), \phi_j(z, \bar{z}) \rangle = \frac{C_{ij}}{(w - z)^{2h} (\bar{w} - \bar{z})^{2\bar{h}}}. \tag{6.112}$$

which vanishes for fields with different conformal dimensions. The coefficients (which are symmetric) are undetermined and can be chosen upon appropriate normalisation of the fields such that $C_{ij} = \delta_{ij}$. This defines a notion of orthogonality between two primary fields. In fact, this can be traced back to the orthogonality of the corresponding states of the Verma Module. Under a global conformal transformation, $w \to \infty, z \to 0$:

$$\lim_{w,\bar{w} \to \infty} w^{2h} \bar{w}^{2\bar{h}} \langle \phi(w, \bar{w}) \phi'(0, 0) \rangle = \langle h, \bar{h} | h', \bar{h}' \rangle \tag{6.113}$$

The three-point function coefficients however remain undetermined.

2. While computing the correlation functions, the strategy of transforming the coordinates to specific points like $\infty, 0, 1$ will be heavily used in order to simplify the problem.

3. Our main strategy to compute the correlation functions between arbitrary primary fields involves the *operator algebra*, which essentially consists of the OPEs of all the primary fields with each other. For example, in order to compute the three-point coefficients, we consider the full OPE of two primary fields:

$$\phi_1(z, \bar{z}) \phi_2(0, 0) = \sum_p \sum_{\{k, \bar{k}\}} C_{12}^{p\{k, \bar{k}\}} z^{h_p - h_1 - h_2 + K} \bar{z}^{\bar{h}_p - \bar{h}_1 - \bar{h}_2 + \bar{K}} \phi_p^{\{k, \bar{k}\}}(0, 0) \tag{6.114}$$

where we have basically expanded the product of two primary fields in terms of all the other primary fields in the theory along with their descendants. $K, \bar{K}$ are the levels of the descendants, i.e. $K = \sum_i k_i$.

4. Taking a correlation of this product with another primary field $\phi_r(w, \bar{w})$ with conformal dimension $\left(h_r, \bar{h}_r\right)$. Transforming $w \to \infty$ on the l.h.s gives,

$$\langle \phi_r \phi_1(z, \bar{z}) \phi_2 \rangle = \lim_{w, \bar{w} \to \infty} w^{2h_r} \bar{w}^{2\bar{h}_r} \langle \phi_r(w, \bar{w}) \phi_1(z, \bar{z}) \phi_2(0, 0) \rangle \tag{6.115}$$

$$= \frac{C_{r12}}{z^{h_1+h_2-h_r} \bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_r}} \tag{6.116}$$

where C_{r12} are the three-point coefficients. On the r.h.s of 6.114 after taking the correlator, it is non-vanishing only for $p = r, \{k, \bar{k}\} = 0, 0$due to the orthogonality of the Verma modules produced by the primary fields. So, we deduce that the coefficients $C_{12}^{p\{0, \bar{0}\}}$ which correspond to the coefficients of the most singular terms in the full OPE are equal to the three-point coefficients.

5. Upon conformal transformation on both sides of equation 6.114, one can deduce the natural separation between the descendant (and holomorphic, anti-holomorphic) coefficients:

$$C_{12}^{p\{k, \bar{k}\}} = C_{12}^{p} \beta_{12}^{p\{k\}} \bar{\beta}_{12}^{p\{\bar{k}\}}. \tag{6.117}$$

The above procedure can explicitly compute these βs in terms of conformal dimensions of all the fields and central charge. We set $\beta_{ij}^{p\{0\}}$ to 1 as a convention.

6. This demonstrates how a full operator algebra for two primary fields can be computed by symmetry once we know the central charge, conformal dimensions of all the fields and the three-point coefficients $C_{\alpha\beta\gamma}$.

6.7 Conformal Blocks, Crossing Symmetry and Conformal Bootstrap

We can use the above operator algebra successively to compute any n-point function. Let's see this in the case of the 4-point correlator,

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle \tag{6.118}$$

1. The anharmonic ratios which are invariant under conformal transformations gives a convenient way to simplify the above problem. We can let $z_1 = \infty, z_2 = 1, z_3 = x, z_4 = 0$ under a global conformal transformation. Essentially, the existence of anharmonic ratio fixes (equals) one of the points when all the other points are mapped to $1, 0, \infty$. The four-point correlator subsequently only depends on this point continuously.

   So the correlator now equals,

$$\lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\langle\phi_1(z_1,\bar{z}_1)\phi_2(1,1)\phi_3(x,\bar{x})\phi_4(0,0)\rangle = G_{34}^{21}(x,\bar{x}) \tag{6.119}$$

$$\equiv \langle h_1,\bar{h}_1|\phi_2(1,1)\phi_3(x,\bar{x})|h_4,\bar{h}_4\rangle \tag{6.120}$$

2. We now use the operator algebra of $\phi_3(z,\bar{z})\phi_4(0,0)$ by separating the p, $\{k,\bar{k}\}$ dependent terms,

$$\phi_3(x,\bar{x})\phi_4(0,0) = \sum_p C_{34}^p x^{h_p-h_3-h_4}\bar{x}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\sum_{\{k,\bar{k}\}}\beta_{34}^{p\{k\}}\bar{\beta}_{34}^{p\{\bar{k}\}}x^K\bar{x}^{\bar{K}}\phi_p^{\{k,\bar{k}\}}(0,0) \tag{6.121}$$

   we can denote the last term which is summed over $\{k,\bar{k}\}$ as $\Psi_p(x,\bar{x}|0,0)$.

3. In this notation, the correlator reads,

$$G_{34}^{21}(x,\bar{x}) = \sum_p C_{34}^p C_{12}^p A_{34}^{21}(p|x,\bar{x}) \tag{6.122}$$

where we define $A^{21}_{34(p|x,\bar{x})}$ as follows (which also decouples into holomorphic and anti- holomorphic parts)

$$A_{34}^{21}(p|x,\bar{x}) = (C_{12}^p)^{-1} x^{h_p-h_3-h_4} \bar{x}^{\bar{h}_p-\bar{h}_3-\bar{h}_4} \langle h_1, \bar{h}_1|\phi_2(1,1)\Psi_p(x,\bar{x}|0,0)|0\rangle \tag{6.123}$$

$$= \mathcal{F}_{34}^{21}(p|x)\bar{\mathcal{F}}_{34}^{21}(p|\bar{x}) \tag{6.124}$$

where

$$\mathcal{F}_{34}^{21}(p|x) = x^{h_p-h_3-h_4} \sum_{\{k\}} \beta_{34}^{p\{k\}} x^K \frac{\langle h_1|\phi_2(1)L_{-k_1}\cdots L_{-k_N}|h_p\rangle}{\langle h_1|\phi_2(1)|h_p\rangle} \tag{6.125}$$

where we have written $(C_{12}^p)^{1/2}$ as $\langle h_1|\phi_2(1)|h_p\rangle$ from the previous section.

4. With this organization of terms, we write the final form of the correlator:

$$G_{34}^{21}(x,\bar{x}) = \sum_p C_{34}^p C_{12}^p \mathcal{F}_{34}^{2l}(p|x)\bar{\mathcal{F}}_{34}^{2l}(p|\bar{x}) \tag{6.126}$$

5. The functions $A_{kl}^{ji}(p|x,\bar{x})$ are called *partial waves* for their analogy with the scattering diagrams used in perturbative QFTs. Using operator algebra between the states $(0,x)$ and $(1,\infty)$ amounts to summing over the intermediate conformal families analogous to the intermediate states formed during the scattering of fields between such states. We can diagrammatically represent this function as follows:

   But note that there is no actual scattering process taking place in CFTs since there is no notion of particles or wave packets separated by some *distance* due to scale invariance.

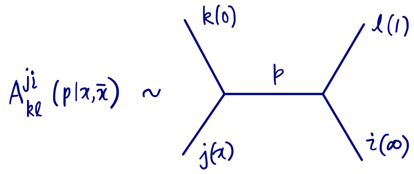

Figure 6.1: Diagrammatic Representation of *partial waves* $A^{ji}_{kl}(p|x, \bar{x}$ analogous to scattering.

6. The functions $\mathcal{F}^{ji}_{kl}$ are called *conformal blocks*. These are the terms of the 4-point functions, which are completely determined by the symmetries. Using the Virasoro algebra, we can compute the numerator of 3.110, which also eventually cancels out the C^p_{12} term from the denominator. The coefficients β are already determined using the operator algebra.

7. The three-point correlators remain the only unknown parts of the correlator. There is another lurking symmetry lying around in this calculation. We could choose to transform the points differently, which amounts to ordering the fields in the correlator in a different way - this is analogous to the crossing symmetry in the scattering terminology. Using this procedure, one can then obtain relations between the different G^{ji}_{kl}s, constraining the unknown three-point coefficients and conformal dimensions. In certain classes of 2D CFTS (minimal models) where there is a finite number of conformal families, this procedure of constraining the parameters by crossing symmetry solves the parameters completely, given the conformal blocks are already computed explicitly by conformal invariance. Such models are thus completely solved. This procedure is called the *Conformal Bootstrap*. For example, consider the above 4-point correlator: we can use global transformations to instead make $z_1 = \infty, z_2 = 0, z_4 = 1$ and z_4 is fixed by the cross-ratio as $1 - x$. This amounts to interchanging the fields ϕ_2 and ϕ_4 in

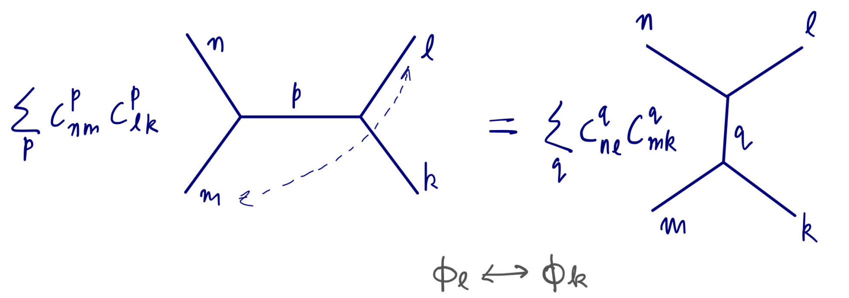

Figure 6.2: Crossing symmetry of 4-point correlators in the diagrammatic language.

the expression6.119 when writing the correlator in terms of G. There is also a change in the argument of G, $x \to 1-x$. So we write,

$$G^{21}_{34}(x,\bar{x}) = G^{41}_{32}(1-x, 1-\bar{x}) \tag{6.127}$$

Similarly if we interchange ϕ_1 and ϕ_4:

$$G^{21}_{34}(x,\bar{x}) = \frac{1}{x^{2h_3}\bar{x}^{2\bar{h}_3}} G^{24}_{31}(1/x, 1/\bar{x}) \tag{6.128}$$

One can express this crossing symmetry in the diagrammatic language again: for example, the constraint 6.127 gives,

$$\sum_p C^p_{21} C^p_{34} \mathcal{F}^{21}_{34}(p|x) \bar{\mathcal{F}}^{21}_{34}(p|\bar{x}) = \sum_q C^q_{41} C^q_{32} \mathcal{F}^{41}_{32}(q|1-x) \bar{\mathcal{F}}^{41}_{32}(q|1-\bar{x}) \tag{6.129}$$

Chapter 7

Virasoro Symmetry in AdS

For an AdS_3, we have a 2-dimensional conformal boundary. And in 2 dimensions, we saw that the conformal group is an infinite-dimensional Virasoro group. One of the key assertions of AdS/CFT correspondence is that the isometry group of AdS is identical to the conformal group on the boundary of AdS. In this short section, we will describe how this identification can be realized between a finite-dimensional isometry group and an infinite-dimensional conformal group in the case of AdS_3. The isometries of AdS_3 are just the group elements of $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. We are familiar with the form of the $SL(2,\mathbb{R})$ elements, these are just

$$f(x) = \frac{ax+b}{cx+d} \in SL(2,\mathbb{R}) \tag{7.1}$$

which have a vanishing Schwarzian derivative,

$$\{f, x^+\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = 0 \tag{7.2}$$

Recall that in the Poincaré coordinates, the metric of AdS_3 reads

$$ds^2 = \frac{L^2}{w^2}\left(-dt^2 + dx^2 + dw^2\right) \tag{7.3}$$

with the light cone coordinate, $x^\pm = x \pm t$, this becomes

$$ds^2 = \frac{L^2}{w^2}\left(dx^+ dx^- + dw^2\right) \tag{7.4}$$

The effect of the isometries on the coordinates can be found to be

$$x^+ \to f(x^+), \tag{7.5}$$

$$x^- \to x^- - \frac{1}{2}w^2\frac{f''}{f'}, \tag{7.6}$$

$$w \to w\sqrt{f'}. \tag{7.7}$$

All the derivatives of f are taken with x^+. The metric then behaves as,

$$ds^2 = \frac{L^2}{w^2 f'}\left(f'dx^+\left[dx^- - wdw\frac{f''}{f'} - \frac{1}{2}w^2\frac{f'''}{f'}dx^- + \frac{1}{2}w^2\left(\frac{f''}{f'}\right)^2 dx^-\right] + \left(dw\sqrt{f'} + \frac{wf''}{2\sqrt{f'}dx^+}\right)^2\right) \tag{7.8}$$

$$= \frac{L^2}{w^2}\left(dx^+dx^- + F(x+)w^2dx^{+2} + dw^2\right) \tag{7.9}$$

with $F(x^+) = -\frac{1}{2}\left\{f, x^+\right\}$, which clearly vanishes for $f \in SL(2,\mathbb{R})$.

Now, if the function f is made arbitrary, i.e. $F(x^+)$ doesn't vanish, we see that the metric 7.9 is modified in the bulk of AdS, but its asymptotic form ($w \to 0$), is still invariant. This shows that the *asymptotic isometry group* of AdS_3 is just the conformal group of the boundary (which is 1+1 dimensional).

So, to summarize:

- The isometries of AdS_3 given by $f \in SL(2,\mathbb{R})$ act on the boundary as the global conformal group of 1+1 Minknowski Space.
- The asymptotic structure-preserving isometries of AdS_3 are given by arbitrary functions f which act on the Bulk via 7.5 - 7.7 and act on the boundary as the local conformal group of 2D Minkowski Space. A further analysis shows that the modes of the generators of the above functions indeed obey the Virasoro algebra. This is expected because we have already seen how arbitrary holomorphic functions in 2 dimensions can be conformal transformations. Using light cone coordinates, the transformation

$x^+ \to f(x^+)$ is indeed analogous to performing conformal transformations on a complex plane.

Chapter 8

The AdS/CFT Correspondence

We now finally connect a theory in AdS to a CFT on the boundary.This can be done in two equivalent prescriptions. Euclidean as well as Lorentzian cases of AdS (AdS/CFT) are discussed.

8.1 The GKP-W Prescription

Discussed first in [11] and in [18], the following correspondence has been proposed in the Euclidean signature of AdS.

1. *CFT defoormations:* The field $\tilde{\phi}_i$ which is obtained via $\phi(x) \to \tilde{\phi}_i(x')$ as the boundary value of bulk fields act as the sources for the boundary theory by coupling to the conformal fields O_i (thus deforming the free theory) via $\int_\partial \phi_\circ O$ adding to the free CFT action. This will basically be fed to the generating functional as the current term.

2. *CFT Generating Functional:* We define the generating functional for the CFT correlators through,

$$\begin{aligned} Z\left(\{\tilde{\phi}_i\}\right) &= \sum_q \frac{1}{q!} \int \prod_{k=1}^{q} d^d x_k \, \langle O_1(x_1) \cdots O_q(x_q) \rangle \, \tilde{\phi}_1(x_1) \cdots \tilde{\phi}_q(x_q) \\ &= \langle \exp \int_\partial \sum_i \tilde{\phi}_i O_i \rangle \end{aligned} \tag{8.1}$$

n-point CFT correlators are then obtained via,

$$\langle O_1 \ldots O_n \rangle = \sum_i \frac{\partial}{\partial \phi_i(x_i)} \cdots \frac{\partial}{\partial \phi_n(x_n)} Z[\{\phi_i\}] \tag{8.2}$$

We get back the free CFT correlators if we take the functional derivatives at $\phi_i = 0$. The deformed theory will be a result of non-trivial boundary sources calculated from the bulk fields.

3. *AdS Generating Functional:* In the bulk we consider the supergravity/string theory partition function with the boundary condition, $\phi \xrightarrow{\infty} \tilde{\phi}$. In the classical approximation of supergravity we take,

$$Z_S[\tilde{\phi}] = \exp\left(-I_S\left(\phi\right)\right) \tag{8.3}$$

where the action $I_S(\phi)$ is computed over the classical solution ϕ obtained by extending the solution $\tilde{\phi}$ on the boundary to the bulk. To do this we first analyse the classical equation of motion for boundary behavior, fix the boundary behavior and compute the bulk-boundary propagator (a Green's function of EOM) to extend the solution. The subtlety in Lorentzian AdS/CFT has to do with the boundary behavior of the bulk fields, as we will see.

4. *The Correspondence:* The coupling hypothesis is already the part of correspondence, the follows stitches the above things together,

$$Z[\tilde{\phi}]_{CFT} = \langle \exp\left(\int_\partial \tilde{\phi}O\right)\rangle = \exp\left(-I_S\left(\tilde{\phi}\right)\right) = Z_S[\tilde{\phi}] \tag{8.4}$$

Using this prescription we now compute free CFT correlators for simple AdS bulk theories.

8.2 Massless Scalar Field

Start with the action of free massless scalar field on AdS_{d+1},

$$I(\phi) = \frac{1}{2}\int_{AdS_{d+1}} d^{d+1}x\sqrt{g}\,\partial_\mu\phi\partial^\mu\phi. \tag{8.5}$$

The equation of motion is,

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\partial^\mu\phi(x)\right) = 0 \tag{8.6}$$

Behavior of Solutions at the boundary:

Claim 8.2.1. *In the case of Euclidean AdS, any function $\phi(\Omega)$ on the boundary of AdS_{d+1} can be uniquely extended to a function $\phi(x)$ in the bulk.*

Proof. Suppose $\phi_1(x)$ and $\phi_2(x)$ be two solutions of equation 8.6 such that their boundary values match. Then $\delta\phi(x) = \phi_1(x) - \phi_2(x)$ vanishes on the boundary and thus an equally good but square integrable solution. □

Bulk-Boundary Propagator:

We now solve for the propagator, $K\ (x, x')$ satisfying 8.6 for all $x \notin \partial$ and behaving like a δ-function when $x \in \partial$ and $x \to x'$. Using this propagator we can then extend the definite boundary solution to the bulk one as,

$$\phi(x) = \int_\partial d^d x' K(x, x')\phi_\circ(x'). \tag{8.7}$$

We will use the poincare coordinates for the Euclidean AdS,

$$ds^2 = \frac{1}{\left(x^0\right)^2}\sum_{\mu=0}^{d}\left(dx^\mu\right)^2. \tag{8.8}$$

Where $x^0 > 0$. the boundary is given by $x^0 = 0$, the point at infinity, $x^0 = \infty$ is a single point ($ds^2 \to 0$). In these coordinates,

$$g_{\mu\nu} = \frac{1}{\left(x^0\right)^2}\delta_{\mu\nu}, \quad g^{\mu\nu} = \left(x^0\right)^2\delta^{\mu\nu}, \quad \sqrt{g} = \frac{1}{\left(x^0\right)^{d+1}}. \tag{8.9}$$

Following Witten's, we will compute the propagator just by symmetry arguments. We start with the boundary point at the infinity, i.e $x'^0 = 0, \bar{x'} \to \infty$. Inheriting the symmetry from the metric and the boundary point, we expect K to be independent of x_i and a function of x^0 only,

$$\partial_0 \left(\frac{1}{\left(x^0\right)^{d+1}} \right) \partial^0 K(x^0) = 0 \tag{8.10}$$

An ansatz of $K(x^0) = c\left(x^0\right)^p$ quickly yields, cx^d as the relevant solution. We can transform the above $K \equiv K\left(x^0, x^i, P\right)$ to a finite x' via an AdS Isometry,

$$x^\mu \longrightarrow z^\mu \equiv \frac{x^\mu}{(x^0)^2 + \vec{x}^2}, \mu = 0, ..., n \tag{8.11}$$

This gives,

$$K(x^0, \vec{x}; P) \longrightarrow K(x^0, \vec{x}; \vec{0}) = c\frac{(x^0)^n}{((x^0)^2 + \vec{x}^2)^n} \tag{8.12}$$

From the translation invariance on the boundary, we get the δ-function at $x^0 = 0, \bar{x} = \bar{x}'$ as,

$$K(x^0, \vec{x}; \vec{x}') = c\frac{(x^0)^n}{((x^0)^2 + (\vec{x} - \vec{x}')^2)^n} \tag{8.13}$$

It is good to note the following result,

Proposition 8.2.2. *A function of the form,*

$$\frac{\epsilon^\beta}{(\epsilon^2 + \vec{x}^2)^\alpha} \tag{8.14}$$

is a δ-function if and only if, $0 < \beta = 2\alpha - d$.

Bulk Solution and the Classical Action:

We thus have,

$$\phi(x^0, \bar{x}) = c \int d^d x' \frac{\left(x^0\right)^d}{\left(\left(x^0\right)^2 + (\bar{x} - \bar{x}')^2\right)^d} \phi_\circ(x'). \tag{8.15}$$

We now evaluate the classical action,

$$
\begin{aligned}
S_{sugra}[\phi(\phi_0)] &= \frac{1}{2}\int d^{d+1}x\sqrt{g}\partial_\mu\phi\partial^\mu\phi \\
&= \int d^{d+1}x\frac{1}{2}\partial_\mu\left(\sqrt{g}\phi\partial^\mu\phi\right) - \frac{1}{2}\int d^{d+1}x\phi\partial_\mu\left(\sqrt{g}\partial^\mu\phi\right) \\
&= \lim_{\epsilon\to 0}\int_{x^0=\epsilon} d^dx\sqrt{g}\phi\partial^0\phi \\
&= -\frac{1}{2}cd\int d^dxd^dx'\frac{\phi_\circ(\bar{x}')\phi_\circ(\bar{x})}{(\bar{x}-\bar{x}')^{2d}}
\end{aligned}
\tag{8.16}
$$

We thus have,

$$
Z[\phi_\circ]_{CFT} = \exp\left(\frac{1}{2}cd\int d^dxd^dx'\frac{\phi_\circ(\bar{x}')\phi_\circ(\bar{x})}{(\bar{x}-\bar{x}')^{2d}}\right) \tag{8.17}
$$

And the free 2-point correlator becomes,

$$
\begin{aligned}
\langle O(\bar{x})O(\bar{x}')\rangle &= \frac{\delta}{\delta\phi(\bar{x})}\frac{\delta}{\delta\phi(\bar{x}')}\left[\exp\left(\frac{1}{2}cd\int d^dyd^dy'\frac{\phi_\circ(\bar{y}')\phi_\circ(\bar{y})}{(\bar{y}-\bar{y}')^{2d}}\right)\right] \\
&\sim \frac{1}{(\bar{x}-\bar{x}')^{2d}}
\end{aligned}
\tag{8.18}
$$

which is precisely the correlator of conformal fields of dimension d. Thus a free scalar field couples to a conformal field of dimension d, which could be directly predicted from the coupling $\int d^dx\phi_\circ O$ for conformal invariance.

8.3 U(1) Gauge Field

The calculation is exactly the same. We note a few critical steps.

The bulk gauge field $A(x^0,\bar{x})$ (a 1-form) are computed with the boundary value $A_\circ(\bar{x}') = a_i(\bar{x}')dx^i$ (a 1-form on the boundary). Starting with the boundary point at infintiy P, we again take the propagator to be dependent only on x^0 in the bulk. If for some $i \geq 1$,

$K^{(i)}(x^0, \bar{x}; P) = f(x^0)dx^i$, we find the functional form of $f(x^0)$ using the EOM. The same symmetry arguments will yield the final form of the bulk-boundary propagator. The bulk solution is then computed by extending each $a_i(\bar{x})$ to the bulk using $K^{(i)}$.

$$A\left(x^0, \bar{x}\right) = \int d^d x' \sum_i K^{(i)} a_i(\bar{x}'). \tag{8.19}$$

The end result is that the $U(1)$ gauge field couples to conformal field of dimensions $d-1$. This also follows from the fact that gauge fields are coordnate independent by defnition and are AdS Isometry scalars. So the component functions of the gauge field have a conformal dimension 1 on the boundary. Thus coupling to $d-1$ dimensional conformal fields.

8.4 Massive Scalar Field

The action for massive scalar field in AdS is,

$$I(\phi) = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi\right). \tag{8.20}$$

Boundary Behavior of solutions:

The boundary behavior here is a bit more subtle compared to the massless case. To analyse the solutions it is fruitful to work in the hyperbolic coordinates. Consider the following metric on AdS,

$$ds^2 = dy^2 + \sinh^2 y d\Omega_d^2 \tag{8.21}$$

where $0 \leq y < \infty$. Note that $det g = \sin g^{2d} y det\gamma$, where γ is the metric on S^d. THe laplacian can be splitted as follows,

$$\Delta = \frac{1}{\sinh^d y}\frac{\partial}{\partial y}\sinh^d y\frac{\partial}{\partial y} - \frac{L^2}{\sinh^2 y} \tag{8.22}$$

where L^2 is the laplacian on the sphere. For large y i.e. on the boundary, the EOM reads,

$$e^{-ny}\frac{d}{dy}\left(e^{ny}\frac{d}{dy}\phi\right) = m^2\phi. \tag{8.23}$$

An ansatz $e^{\lambda y}$ works as the solution iff, $\lambda\,(\lambda + d) = m^2$. With λ_+, λ_- being the solutions of the quadratic equation, the behavior of the field ϕ near the boundary is dominated by $e^{\lambda_+ y} = (e^{-y})^{-\lambda_+}$, the contribution from the greater root.

Proposition 8.4.1. *e^{-y} was an arbitrary ansatz. For any function f with 1st order zero (simple zero) on the boundary, $\phi(y, \bar{x}) \sim \left(f(y)\right)^{-\lambda_+}\phi_\circ(\bar{x})$ can be taken as the boundary behavior.*

But note that the metric of the boundary is not fixed, there is a conformal class of metrics. It can be induced by again a function with simple zero on the boundary (multiplied to the bulk metric as a conformal factor). We thus have the following consequence,

Corollary 8.4.2. *The function $\phi_\circ(\bar{x})$ (the coefficient of a non-normalizable mode) is then arbitrary due to the freedom in f i.e $f \to e^{w(\bar{x})f}$. It's a conformal field of dimension $-\lambda_+$ coupling to aconformal field of dimension $d + \lambda_+$ in the boundary CFT.*

This follows from noting the behavior of conformal fields under a conformal transformation, $ds^2 \to \Lambda^2 ds^2 \implies \phi \to \Lambda^{\frac{\Delta}{2}}\phi$. We then follow the same procedure as before to compute the bulk-boundary propagator and the bulk field respecting the above boundary behavior. We end up with the result that the massive scalar fields couple to conformal fields O_Δ with conformal dimension,

$$\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right) \tag{8.24}$$

We can generalize these discussions to other fields as well, say for any $p-form$. Massless $p-forms$ have component fields with conformal dimension p, thus coupling to conformal operators of dimension $\Delta = d - p$. A massive p-form would couple to an operator with dimension $\Delta = d + \lambda_+ - p$, where λ_+ is the greater solution of $\lambda(d+\lambda) = m^2$. So, $(\Delta - d + p)(\Delta + p) = m^2$,

$$\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2 - 4dp + p^2}\right). \tag{8.25}$$

8.5 Lorentzian AdS/CFT

In this section, we will briefly summarize the work of [3] on Lorentzian AdS/CFT correspondence. In Euclidean case we were able to uniquely extend the boundary solution to the bulk, and then computed the classical action which gave away the CFT correlators through the correspondence 8.4. In Lorentzian signature, the important fact is that there exist solutions of the wave equations which are normalizable and thus which fall off faster near the boundary than the non-normalizable modes and thus not effecting the boundary behavior. This creates an ambiguity in the bulk solution through which we relate to the boundary. The major claim of the work cited is that the non-normalizable solutions are to be seen as the non-fluctuating modes which dictate the boundary behavior, and importantly which couple to the CFT operators - deforming the boundary theory. The normalizable modes on the other hand are the fluctuaing modes in the bulk - which could be quantized. These correspond to the states in the CFT as well.

Proposition 8.5.1. *The solutions of wave equations in Lorentzian signature have vast possibilities of solutions. We extract the fluctuating solution to relate to the states in the boundary theory and the non-fluctiating mode as the boundary behavior which couple to the CFT operators. In particular non-flucutaing/non-normalizable modes correspon to the deformations of the CFT theory and the normalizable modes contribute to the VEV of the CFT operators due to an excited state.*

We can illustrate this in the scalar field case as follows

In Euclidean case the bulk solution read,

$$\phi(x^0, \bar{x}) = \int_\partial K\left(x^0, \bar{x}, \bar{x}'\right) \phi_b\left(\bar{x}'\right) \tag{8.26}$$

In the Lorentzian case there is also a normalizable mode,

$$\phi = \phi_n + \int_\partial K \phi_b = \phi_n + \phi_{nn} \tag{8.27}$$

We can compute the classical action (8.16). Then the relevant term that contributes to the 1 point function (say) is given by the functional derivative $\frac{\partial S}{\partial \phi} = \left(x^0\right)^{-d+1} \frac{\partial \phi}{\partial x^0}$, which from above equation 8.27 reads (taking the boundary limit $\phi_n \to (x^0)^d \tilde{\phi}_n$),

$$\frac{\partial \phi}{\partial x^0} = d(x^0)^{d-1} \tilde{\phi}_n(\bar{x}) + cd(x^0)^{d-1} \int d^d x' \frac{\phi_b(\bar{x}')}{(\bar{x} - \bar{x}')^{2d}} \tag{8.28}$$

So,

$$\langle \tilde{\phi}_n \mid O | \tilde{\phi}_n \rangle = d\tilde{\phi}_n(\bar{x}) + cd \int d^d x' \frac{\phi_b(\bar{x}')}{(\bar{x} - \bar{x}')^{2d}} \tag{8.29}$$

We see that there is an additional contribution from the normalizable mode.

Proposition 8.5.2. *The boundary behaviour of the classical EOM of a scalar field ϕ in the AdS as,*

$$\phi \sim \alpha(x^0)^{d-\Delta} + \beta(x^0)^\Delta \tag{8.30}$$

in terms of the non-normalizable (α piece) and the normalizable (β piece) modes. Then prescription of correspondence in the lorentizan AdS is that, non-normalizable modes are mapped to the sources on the boundary deforming the CFT Hamiltonian as,

$$H = H_{CFT} + \alpha O \tag{8.31}$$

and the normalizable modes are mapped to the VEVs or states of the boundary CFT,

$$\langle \beta | O | \beta \rangle \sim \beta + (\alpha\ contribution) \tag{8.32}$$

Turning off the normalizable mode, we get the VEV of the dual operator and a non-trivial mode maps to an excited state on the boundary. The map from the bulk normalizable (fluctuating) modes to the states in the boundary CFT can be explicilty realized by looking at the representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ which is the conformal algebra of the boundary as well the isometry algebra of the bulk. We find that the normalziable modes can be realised as the unitary representations where the non-nromalizable modes as the non-unitary representations. The details of behavior of normalizable and non-normalizable solutions near the boundary is explictly studied in the reference along with their belonging to different representations.

8.6 BDHM Dictionary

There is a second dictionary for the correspondence. Instead of letting the sources couple to the boundary CFT operators and computing the correlations via 8.4, we can extrapolate the correlation functions of bulk fields to the boundary by suitably compensating the decay behavior of the bulk fields.

For discussing a theory at infinity, we want to take a $x \to \partial$ limit to the AdS field $\phi(x)$. And for a normalizable field, we know that $\phi \to 0$ as $x \to \infty$. But note that when there are finite energies involved in a QFT, the fields have a universal behavior near infinity. For example, the massless fields fall off as $\frac{1}{r}$ and the massive ones decay like e^{-mr}. A finite non-vanishing field at infinity can then be obtained from these fields by compensating the decay with a multiplicative factor and then taking the asymptotic limit. This is already clear from the discussion on the behavior of fields in Lorentzian AdS (see 8.30).

Recall the discussion on approaching the boundary from section 3.6. Consider Euclidean AdS and approach the boundary as,

$$\rho\left(\epsilon,t\right)=\frac{\pi}{2}-\epsilon e^{-t} \tag{8.33}$$

which gives a Euclidean Flat space metric on the boundary. If $\phi(x)$ is the field of a scalar particle in AdS, we define the operator dual to ϕ on the boundary as,

$$O\left(t,\Omega\right)=\lim_{\epsilon\to 0}\frac{\phi(t,\rho(\epsilon,t),\Omega}{\epsilon^{\Delta}} \tag{8.34}$$

where Δ is the power involved in the boundary behavior of field ϕ. The power Δ ensures the conformal symmetry (AdS Isometry) is respected. Let's call this operator, *holographic dual to* ϕ.

Proposition 8.6.1. *Holographic dual Operator* $O\left(t,\Omega\right)$ *has the correlators of a CFT on the Euclidean space.*

One can consider a massive/massless scalar theory in AdS to check this. In that case, it is possible to explicitly solve for the fields using hypergeometric functions. We expect that under the boundary limit, factored with a suitable conformal dimension, the calculation for the CFT correlators simplifies drastically via the bulk theory.

So, the alternative prescription for AdS/CFT correspondence is that,

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle=\lim_{\epsilon\to 0}\epsilon^{-n\Delta}\left\langle\phi_1(x_1)\cdots\phi_n(x_n)\right\rangle \tag{8.35}$$

where $\phi_1,\ldots,\phi_n$ are bulk AdS fields corresponding to a gravity theory with characteristic conformal dimension Δ.

Chapter 9

Conclusion: AdS/CFT and Geometry

We are now equipped with the basic conformal invariants, $tr(W \otimes \cdots \otimes W)$. The procedure of finding other conformal invariants is, however, not yet clear. In the Riemannian case, due to a classical theorem from Weyl (ala Ricci Calculus where we take *covariant* derivatives), any Riemannian invariant can be written as a combination of $tr\ (\nabla R \otimes \cdots \otimes\ \nabla R)$.

Question 9.0.1. *How to produce conformal invariants? Can we write an elementary formula like in the Riemannian case that gives us all the conformal invariants?*

This question was partially answered by Fefferman and Graham in [9]. In doing so, they have laid out a foundational theory for conformal geometry and also AdS/CFT which was yet to be born. We will only sketch the ideas of these developments and connections in this chapter.

Consider a Riemannian manifold (M, g). We formally define (scalar, local) conformal invariants as polynomials in g_{ij} and $\partial_\alpha g_{ij}$ satisfying the following:

1. If g and g' are isometric, then $P(g) = P(g')$.

2. If $g = \lambda(x)g'$ for a smooth positive function λ, then $P(g) = \lambda^{\mathrm{w}} P(g')$ for some power or weight w.

This is reminiscent of the conformal fields in a CFT! We indeed wanted fields symmetric/covariant (or as we define above with weights - *invariant*) under conformal transformations.

Conformal boundary or conformal infinity can also be formally defined as:

Definition 9.0.1 (Conformal Infinity). *Let M be a manifold with boundary and suppose that a conformal structure $[g]$ is given on ∂M. Let $f \in C^\infty(\bar{M}$ and $f > 0$ in $M, f = 0, df \neq 0$ in ∂M. Then a Riemannian metric g^+ on $M \sim \partial M$ is said to have $[g]$ as conformal infinity if for some number $k > 0$, $f^k g^+$ has a smooth extension to $\bar{M}$, and when restricted to $T\partial M$, $f^k g^+ \in [g]$. This is independent of the choice of the function f.*

9.1 The Ambient Metric

The work of [9] was motivated by the following matching between the symmetry groups of manifolds.

Motivation. *The isometries of Lorentzian $R^{n+1,1}$ act conformally on the Sphere S^n: Riemann invariants of $R^{n+1,1}$ give the conformal invariants of S^n.*

This is because one can re-write sphere in projective coordinates ,

$$G : \sum_{i=1}^{n+1} x_i^2 - 1 = 0 \xrightarrow{x^k = \frac{\xi_k}{\xi_0}} \tilde{G} : \sum_{i=1}^{n+1} \xi_k^2 - \xi_0^2 = 0.$$

The metric on $\tilde{G}$ of the Lorentzian type,

$$\tilde{g} = \sum_{i=1}^{n+1} d\xi_k^2 - d\xi_0^2$$

restricts to the sphere G as,

$$\tilde{g}_G = \xi_0^2 \sum_{i=1}^{n+1} dx_i^2.$$

Clearly, the metric on S^n is affected by a conformal factor when the metric $\tilde{g}$ is preserved by the isometries of $\tilde{G}$. FG showed that one can always construct such an *ambient* metric - the restriction of which onto the conformal manifold is unique and given by a formal taylor-series like expansion. It is also shown that, not only an ambient Lorentz type, but equivalently an ambient Poincaré metric on a manifold of one dimension higher can be constructed whose isometries again act conformally on the embedded manifold. This is precisely the situation in AdS. The isometries of the bulk act conformally on the boundary manifold which has a conformal structure. So, one gets the conformal invariants for the boundary metric by constructing Riemann invariants of an ambient/bulk metric. We state here the result for Poincaré type ambient metric which is directly related to AdS/CFT.

Theorem 9.1.1 (Fefferman-Graham)**.** *Let M be a n-manifold with conformal structure $[g]$. And $M^+ = M \times [0,1]$ where $M \sim \partial M^+ = M \times \{0\}$. A metric g^+ satisfying*

a) g^+ has [g] as conformal infinity,

b) $Ric(g^+) = -ng^+$

can be written in certain coordinates $(x^1, \ldots, x^n, r)$ as:

$$g^+ = r^{-2}\left[dr^2 + g^+_{ij}(x,r)dx^i dx^j\right],$$

where r defines $M \subset M^+$ and $(x^1, \ldots, x^n)$ forms coordinates on M. Along with (a), (b) if we further ask for: (c) g^+_{ij} to be an even function with respect to variable r, then we have the following required results.

1. *n odd. Up to a diffeomorphism fixing M, there is a unique formal power series solution g^+ to a)-c). If $[g]$ is real analytic, then the power series converges so that g^+ exists and $r^2 g^+$ is analytic up to the boundary; written generally as,*

$$g_r \sim g_{(0)} + r^2 g_{(2)} + \cdots + r^{n-1} g_{(n-1)} + r^n g_{(n)} + r^{n+1} g_{(n+1)} + \ldots$$

2. *n even. There are conformal structures for which there is no formal power solution of a)-c). However, if b) is replaced by: b') Along M, the components of $Ric(g^+) + ng^+$ vanish to order $n-2$, then there is a formal power series solution for g^+ uniquely determined*

up to addition of terms vanishing to order $n-2$ and up to a diffeomorphism fixing M; written generally as,

$$g_r \sim g_{(0)} + r^2 g_{(2)} + \cdots + r^{n-2} g_{(n-2)} + r^n g_{(n)} + r^n \log r\, h + r^{n+1} g_{(n+1)} + \ldots$$

Remark. *Einstein's equation fixes the power m related to the conformal infinity. There is a coordinate system in which the metric g^+ is Poincaré type. The normalisation n in (b) is a choice of convenience, and we are free to choose otherwise.*

The formal power series can be explicitly computed using Einstein's equations using a representative boundary metric $\gamma \in [g]$. Using this it has been proved that in odd dimensions, all local scalar conformal invariants can be computed. It has also been extended to even dimensions after a little more work.

9.2 Connection to AdS/CFT

In the context of AdS/CFT, one uses this expansion to reconstruct the bulk (asymptotically) AdS space. The log term in even dimensions is seen to be related to the conformal anomaly of the CFT living on the boundary [5], which vanishes in odd dimensions! This allowed for a holographic calculation of conformal anomalies of CFTs. The idea is to consider a bulk action of gravity, including certain boundary and counter terms. Using the Fefferman-Graham expansion of the bulk metric, and upon a suitable regularisation (and renormalisation) procedure, the conformal anomaly was calculated [12]. On the other side of developments. The conformal anomalies of CFTs in arbitrary dimensions are proposed to be precisely the conformal invariants for the given conformal structure [6].

Proposition 9.2.1 (Desser-Schwimmer)**.** *The conformal anomaly of a CFT living on a manifold of dimension d is given by a type A term, which is a topological invariant (Euler density* *in particular), and a type B term, which is a conformal invariant:*

$$\mathcal{A} \sim E_{(d)} + I_{(d)}$$

All other terms are local and can be cancelled using appropriate counter terms.

In proving this, Desser and Schwimmer conjectured a certain global decomposition of conformal invariants, which, using the language of Ambient metric, was proved in a series of papers (and a book) in mathematics literature starting from [1] with a complete proof finally in 2012.

9.3 (Further) Directions

We end with a summary of the directions discussed in this thesis.

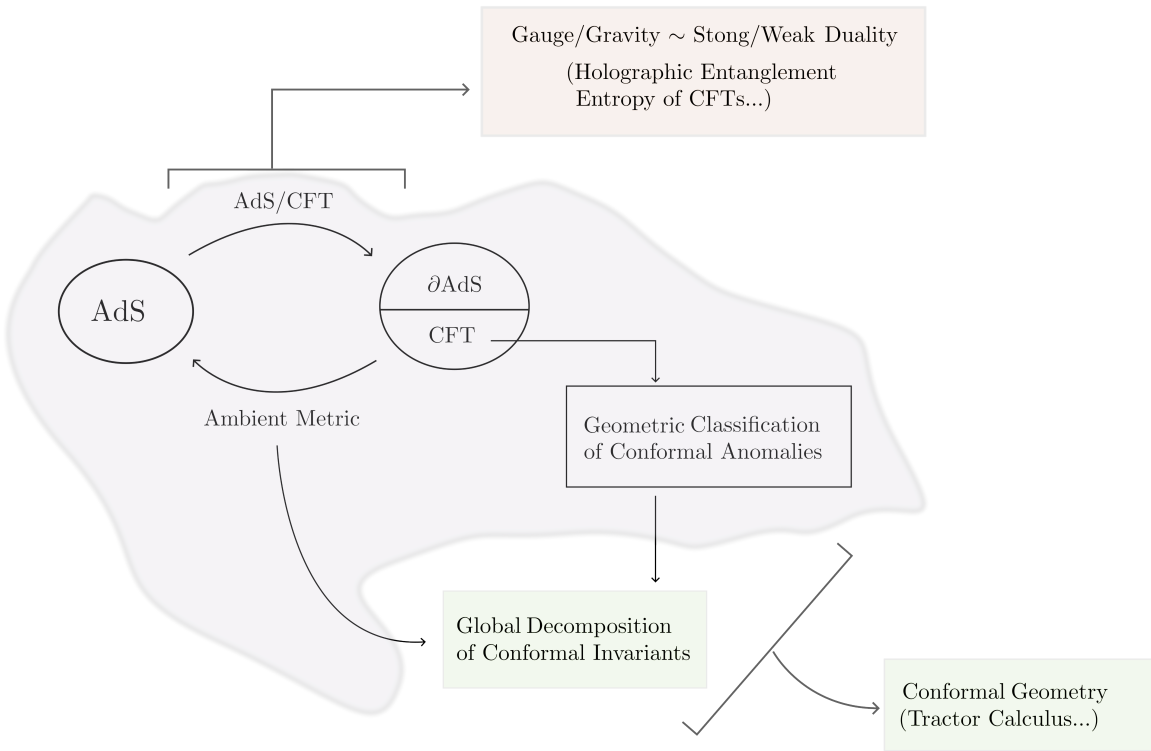

Figure 9.1: AdS/CFT and (Conformal) Geometry

From the physics point of view, AdS/CFT correspondence is a powerful tool to convert calculations on CFT, say, entanglement entropy. The basic (holographic) idea is to consider different kinds of perturbations to the bulk theory, viz., perturbations to the bulk metric and through the two prescription deuce how these perturbations couple/define the conformal operators on the boundary.

For example, consider the following general scalar, vector and tensor perturbation to the AdS_3,

$$ds^2 = U^2[-dt^2 + (dx)^2]\frac{dU^2}{U^2} + d\Omega_3^2 \\ + \left[\frac{T}{U^{n_t}}(dt - dx)^2 + \frac{V}{U^{n_v}}(dt - dx)d\theta + \frac{S}{U^{n_s}}d\theta^2\right].$$

Note that, $U = \frac{1}{x^0}$ gives back the Poincaré patch coordinates on the free AdS. From the discussion in section 8.4 we note that the massive scalar fields in AdS_3 couple to a conformal field of dimension $\Delta = 1 + \sqrt{1 + m^2}$. Similarly by a suitable analysis of vector and tensor fields starting from their actions, one finds the corresponding operators of CFT_2. Using this setup one can ask many questions about a CFT in lower dimensions, by perturbating the bulk theory appropriately. For example, one can compute novel things using the framework of Holographic Entanglement Entropy introduced in [17].

From the point of view of Mathematics, conformal geometry is a rich field of research. There have been large efforts to develop a more nataural way to do it. Like Ricci calculus and Riemann Connections, there is a conformally covariant way to take the derivatives via tractor[1] connections. Knowing the invariants of a structure is a preliminary and crucial step to understand the structure further. To produce conformal invariants (scalars, operators…) more naturally using the tractor calculus has been a major focus of conformal geometry. There is a notion of Q-curvataure which is anologous to the scalar Riemannian curvature. More broadly conformal geometry falls into a larger study of parabolic geometry and cartan

[1]Tractor is a portmanteau of Tracy Thomas and twistor.

connections. Some of the current problems include generalization of the Yamabe problem, which asserts the existence of a constant scalar curvature metric that is conformal to a given metric. It occurs in the attempts to formulate higher dimensional uniformization theorem using constant scalar curvataure metrics. Conformal Geometry also plays an important role in twistor theory approach to quantum gravity.

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Appendix A

Classical Symmetries in QFT

We discuss a general Theory of Symmetries and Conservation Laws in QFT. Consider a general action:

$$S = \int d^d x \mathcal{L} \left(\phi, \partial_\mu \phi\right) \tag{A.1}$$

We study the transformations affecting both positions and fields (active point of view):

$$x \longrightarrow x' \tag{A.2}$$

$$\phi(x) \longrightarrow \phi' \left(x'\right) \equiv \mathcal{F} \left(\phi(x)\right) \tag{A.3}$$

Under this transformation, the action reads

$$S' = \int d^d x' \mathcal{L} \left(\phi' \left(x'\right), \partial'_\mu \phi'(x')\right) \tag{A.4}$$

$$= \int d^d x \mid \frac{\partial x'}{\partial x} \mid \mathcal{L} \left(\mathcal{F}(\phi(x)), \left(\frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x))\right)\right) \tag{A.5}$$

Examples:

1. *Translation:*

$$x' = x + a \tag{A.6}$$

$$\phi'(x + a) = \phi(x) \tag{A.7}$$

$$S' = S. \tag{A.8}$$

2. *Lorentz Transformation:*

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{A.9}$$

$$\phi'(\Lambda x) = L_{\Lambda}\phi(x) \tag{A.10}$$

L_{Λ}, a matrix representation of Lorentz Group on the space of fields, acts on the components of ϕ.

$$S' = \int d^d x \mathcal{L}\left(L_{\Lambda}\phi, \Lambda^{-1}\partial\left(L_{\Lambda}\phi\right)\right) \tag{A.11}$$

Λ as a matrix satisfies $\Lambda^T \eta \Lambda = \eta$.

For $L_{\Lambda} = 1$ (Scalar Fields): $S' = S$ if ∂_{μ} appears in a Lorentz invariant way in the Lagrangian. Allowing almost two derivatives in the theory gives the following most general form of Lagrangian.

$$\mathcal{L}\left(\phi, \partial_{\mu}\phi\right) = f\left(\phi\right) + g\left(\phi\right)\partial_{\mu}\phi\partial^{\mu}\phi \tag{A.12}$$

3. *Sclae Transformation:*

$$x' = \lambda x \tag{A.13}$$

$$\phi'\left(\lambda x\right) = \lambda^{\Delta}\phi\left(x\right) \tag{A.14}$$

Where Δ is called the *scaling dimension* of the field ϕ.

$$S' = \lambda^d \int d^d x \mathcal{L}\left(\lambda^{-\Delta}, \lambda^{-1-\Delta}\partial_{\mu}\phi\right) \tag{A.15}$$

So, for a (massless) scalar field:

$$S[\phi] = \int d^d x \partial_\mu \phi \partial^\mu \phi \tag{A.16}$$

$S' = S$ if $\Delta = \frac{d}{2} - 1$. We are free to add any ϕ^n as long as $\Delta n = d \implies n = \frac{2d}{(d-2)}$, preserving the scale invariance. *The only possibilities for n being even and 'ensuring stability' are* $n = 6$ when $d = 3$ and $n = 4$ when d = 4 i.e. ϕ^6 and ϕ^4.

4. Various transformations may be defined that only effect the field ϕ and not the coordinates. For example:

$$\phi'(x) = e^{i\theta}\phi(x) \tag{A.17}$$

for a complex field. Or, more generally even,

$$\phi'(x) = R_\omega \phi(x) \tag{A.18}$$

where R_ω is some representation of a Lie Group parametrized by the group coordinate ω.

A.1 Infinitesimal Transformations

For any general infinitesimal transformation:

$$x'\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \tag{A.19}$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \tag{A.20}$$

$\{\omega_a\}$ - set of infinitesimal parameters.

Generator of infinitesimal transformations (of the field) - G_a:

$$-i\omega_a G_a \phi(x) := \delta_\omega \phi(x) = \phi'(x) - \phi(x) \tag{A.21}$$

$$i\omega_a G_a \phi = \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi - \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{A.22}$$

Examples:

1. *Translation:*
$$\frac{\delta x^{\mu}}{\delta \omega_{\nu}} = \delta^{\mu}_{\nu}; \; \frac{\delta \mathcal{F}}{\delta \omega^{\nu}} = 0. \tag{A.23}$$

   We thus have the generator for translations:
$$P_{\mu} = -i\partial_{\mu}. \tag{A.24}$$

2. *Lorentz Transformation:* Using $\Lambda^{\mu}_{\nu} = x^{\mu} + \omega^{\mu}_{\nu} x^{\nu}$, $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ in equation A.11,
$$x'^{\mu} = x^{\mu} + \omega^{\mu}_{\nu} x^{\nu} \tag{A.25}$$

   Expand using the basis of antisymmetric matrices $M^{\rho\sigma}$ (satisfies the Lorentz Lie Algebra),
$$\omega^{\mu}_{\nu} = \frac{1}{2}\Omega_{\rho\sigma}\left((M^{\rho\sigma})^{\mu}_{\nu}\right) \tag{A.26}$$

   Then,
$$\begin{aligned}
\delta x^{\mu} &= \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}_{\nu} x^{\nu} && \text{(A.27)}\\
&= \frac{1}{2}\Omega_{\rho\sigma}\left(\eta^{\rho\mu}\delta^{\sigma}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}_{\nu}\right) x^{\nu}\\
&= \frac{1}{2}\Omega_{\rho\sigma}\left(\eta^{\rho\mu}x^{\sigma} - \eta^{\sigma\mu}x^{\rho}\right)\\
\frac{\delta x^{\mu}}{\delta \Omega_{\rho\sigma}} &= \frac{1}{2}\left(\eta^{\rho\mu}x^{\sigma} - \eta^{\sigma\mu}x^{\rho}\right). && \text{(A.28)}
\end{aligned}$$

   Also, write,
$$\mathcal{F}(\phi) = L_{\Lambda}\phi \approx 1 - \frac{1}{2} i\Omega_{\rho\nu}S^{\rho\nu}. \tag{A.29}$$

   Where $S^{\rho\nu}$ is the generator of Lorentz algebra on the space of fields. Basically, it generates a Lorentz representation L_{Λ} on the space of fields, satisfying the Lorentz Algebra.

We thus have,

$$
\begin{aligned}
\frac{1}{2} i \Omega_{\rho\nu} L^{\rho\nu} &= \frac{1}{2} \Omega_{\rho\nu} \left(x^{\nu} \partial^{\rho} - x^{\rho} \partial^{\nu} \right) \phi + \frac{1}{2} i \Omega_{\rho\nu} S^{\rho\nu} \phi \\
L^{\rho\nu} &= i \left(x^{\rho} \partial^{\nu} - x^{\nu} \partial^{\rho} \right) + S^{\rho\nu}.
\end{aligned} \tag{A.30}
$$

A.2 Noether's Thoorem

Action under an infinitesimal transformation is invariant only if parameters ω_a are x-independent. "Rigid Transformations".

Upto first order in ω_a note the following:

$$
\begin{aligned}
\frac{\partial x'^{\nu}}{\partial x^{\mu}} &= \delta^{\nu}_{\mu} + \partial_{\mu} \left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a} \right) \\
\left| \frac{\partial x'^{\nu}}{\partial x^{\mu}} \right| &= 1 + \partial_{\mu} \left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \right) \\
\frac{\partial x^{\nu}}{\partial x'^{\mu}} &= \delta^{\nu}_{\mu} - \partial_{\mu} \left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a} \right)
\end{aligned} \tag{A.31}
$$

Then,

$$
S' = \int d^d x \left(1 + \partial_{\mu} \left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \right) \right) \tag{A.32}
$$

$$
\times \mathcal{L} \left(\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta^{\nu}_{\mu} - \partial_{\mu} \left(\omega_a \left(\delta x^{\nu} / \delta \omega_a \right) \right) \right] \left(\partial_{\nu} \Phi + \partial_{\nu} \left[\omega_a \left(\delta \mathcal{F} / \delta \omega_a \right) \right] \right) \right) \tag{A.33}
$$

Now, the terms of $\delta S = S' - S$ containing no derivatives of ω_a sum up to zero if the action is symmetric under rigid transformation. And the terms containing the first derivative of ω_a remain.

$$
\delta S = - \int dx \, j^{\mu}_{a} \partial_{\mu} \omega_a \tag{A.34}
$$

with,

$$j_a^\mu = \left\{ \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta_\nu^\mu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{A.35}$$

This is the *current* associated with the infinitesimal transformation A.23, A.24.

Integration by parts gives,

$$\delta S = \int d^d x \left(\partial_\mu j_a^\mu\right) \omega_a \tag{A.36}$$

For an on-shell configuration, we have $\delta S = 0$, so

$$\partial_\mu j_a^\mu = 0 \tag{A.37}$$

since, ω_a is arbitrary.

Thus every continuous symmetry $\implies$ $\exists$ conserved current (canonical) A.35.

Conserved charge associated to j_a^μ:

$$Q_a = \int d^{d-1} x \, j_a^0. \tag{A.38}$$

This is all a classical result, and says little about Quantum realization of these symmetries...

Energy Momentum Tensor - The GR way

Consider 'general' infinitesimal coordinate transformation:

$$x'^\mu = x^\mu + \epsilon^\mu(x) \tag{A.39}$$

We identify the stress tensor through,

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu \tag{A.40}$$

Assuming $T^{\mu\nu}$ to be identically symmetric,

$$\delta S = \int d^d x T^{\mu\nu} \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right). \tag{A.41}$$

Now considering $x' = x + \epsilon$ as an infinitesimal coordinate transformation on a pseudo-Riemannian manifold with metric $g_{\mu\nu}$, then

$$\begin{aligned} g'^{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= \left(\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha \right) \left(\delta^\beta_\nu - \partial_\nu \epsilon^\beta \right) g_{\alpha\beta} \\ &= g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right) \end{aligned} \tag{A.42}$$

So,

$$\delta S = -\frac{1}{2} \int d^d x T^{\mu\nu} \delta g_{\mu\nu}. \tag{A.43}$$

We thus have an alternative definition for $T^{\mu\nu}$ as the functional derivative of the action w.r.t metric, evaluated at any given space-time!

Consider Scalar field theory on an arbitrary manifold, for example.

$$S = \int d^d x \sqrt{g} \mathcal{L} \tag{A.44}$$

$$= \frac{1}{2} \int d^d x \sqrt{g} \left\{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right\} \tag{A.45}$$

Now, use the following hefty stuff:

$$det A = e^{tr \ln A}, \; \delta g^{\mu\nu} = -g^{\alpha\mu} g^{\beta\nu} g_{\alpha\beta}, \; \delta \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} \tag{A.46}$$

And we find that

$$T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \partial^\mu \phi \partial^\nu \phi. \tag{A.47}$$

The Upshot of all this is we get an identically symmetric stress tensor! The downside of all this is that the calculation can be involved/cumbersome.

A.3 Rotation Algebra

Recall, the rotation and Lorentz (generalized rotations) algebras.

Rotations are just those transformations that preserve a given norm.

Consider usual euclidean rotations, that is $dx^2 = \sum dx_i^2$.

Rotations:

$$dx' = Rdx \text{ such that } dx'^2 = dx^2, det(R) = 1 \tag{A.48}$$

Rotations are just $SO\,(D)$. Let's find out the infinitesimal generators of this transformation.

$$R = I + \epsilon D \tag{A.49}$$

D turns out to be anti-symmetric using A.48. Such a matrix is simply generated (a linear combination of) by

$$J^{ij}_{(mn)} = -i\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right) \tag{A.50}$$

putting those 1s, -1s appropriately across at $D \times D$ Let $R_1 = I + A$, $R_2 = I + N$.

Then,

$$\begin{aligned} R_1 R_2 &= I + A + B + AB \\ R_2 R_1 &= I + A + B + Ba \\ (R_2 R_1)^{-1} &= I + [A, B] \end{aligned} \tag{A.51}$$

This gives us a reason to calculate the commutations :). It suffices to calculate commutation between the generators.

Now using equation A.50 it is easy to show the $SO\,(D)$ Algebra.

$$\left[J_{(mn)}, J_{(pq)}\right] = i\left(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - (p \leftrightarrow q)\right) \tag{A.52}$$

Consider the generalized rotations. These basically form $SO\,(m, n)$, which preserves the norm - $ds^2 = \sum^m i = 1\,(dx_i)^2 - \sum^{m+n} i = m + 1\,(dx_i)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. $det(R) = 1$.

$$R = I + i\Theta^{\mu\nu} J_{\mu\nu} \tag{A.53}$$

with

$$J^{\mu\nu(\rho\sigma)} = -i\left(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}\right). \tag{A.54}$$

We then have the $SO(m, n)$ Lie Algebra as,

$$\left[J_{(mn)}, J_{(pq)}\right] = i\left(\eta_{mp}J_{(nq)} + \eta_{nq}J_{(mp)} - (p \leftrightarrow q)\right) \tag{A.55}$$

And the Poincaré Algebra is then evaluated by supplementing $J_{\mu\nu}$ with $P_\mu = i\partial_\mu$.

$$\left[P_\mu, P_\nu\right] = 0 \tag{A.56}$$

$$\left[J_{\mu\nu}, P_\rho\right] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \tag{A.57}$$

$$\left[J_{(mn)}, J_{(pq)}\right] = i\left(\eta_{mp}J_{(nq)} + \eta_{nq}J_{(mp)} - (p \leftrightarrow q)\right) \tag{A.58}$$

A.4 On 'exponentiating' infinitesimal transformations

General Theory: An infinitesimal transformation of any generic field $f(x)$ can be considered as a 1-parameter variation whose infinitesimal form is known to be

$$f(\epsilon)(x) = f(0)(x) + \epsilon G(x) + O(\epsilon^2) \tag{A.59}$$

For a field $\phi(x)$ in classical field theory, $G(x) \sim \tilde{G}\phi(x)$, where $\tilde{G}$ is the usual infinitesimal generator A.21 for a given transformation. The finite effect of the transformation is then given by the integral curve,

$$\dot{\phi}(\epsilon)(x) = G(x) \tag{A.60}$$

And for a field theory, we have

$$\frac{d\phi_\epsilon(x)}{d\epsilon} = -iG\phi(x) \tag{A.61}$$

which defines the finite flow using the infinitesimal information viz. 'Integral Curves'!

Exponentiating the Special Conformal Transformation:

Consider the $\epsilon-$variation of x:

$$x^\mu(\epsilon) = x + \epsilon\xi^\mu$$

We know ξ^μ for the special conformal transformation from table 4.1. We will now derive the finite form given in the same table.

We have,

$$x'^\mu = x^\mu + \epsilon 2(x \cdot b)x^\mu - b^\mu x^2 \tag{A.62}$$

$$x_\epsilon = x_0 + \epsilon\left(2(b.x)x - bx^2\right) \tag{A.63}$$

So, we want to solve the integral curve,

$$\dot{x}_\epsilon = 2\left(b.x_\epsilon\right)x_\epsilon - b\epsilon^2 \tag{A.64}$$

We use a magical change of variables:

$$y = \frac{x_\epsilon}{x_\epsilon^2} \tag{A.65}$$

This gives us,

$$\dot{y} = -b \implies y = y_0 - \epsilon b \tag{A.66}$$

We can include the 'extent of flow' parameter ϵ in the strength of b i.e. $\mid b \mid$ itself. Which gives,

$$y = y_\circ - b \tag{A.67}$$

From here on, we will drop the ϵ notation on x_ϵ. And continue to write the initial x to be $x_\circ$ Then,

$$\frac{x}{x^2} = \frac{x_\circ}{x_\circ^2} - b \tag{A.68}$$

$$\tag{A.69}$$

Note that,

$$\frac{x}{x^2} = A \implies x = \frac{A}{A^2}.$$

We thus have,

$$x = \frac{x_\circ - bx_\circ^2}{1 + x_\circ^2 b^2 - 2(b \cdot x_\circ)} \tag{A.70}$$

Scaling Dimension and $\tilde{\Delta}$: Knowing that $\tilde{\Delta} \sim I$, we let $\tilde{\Delta} = -i\Delta I$ connect the scaling dimension of the field with that of the scaling generator. We will show that this is right. It's almost trivial at this point, but let's write things down explicitly anyway.

For scaling, we have a simple ϵ - variation,

$$x(\epsilon) = 1 + \epsilon x \tag{A.71}$$

Then, the finite flow is generated as,

$$\frac{dx(\epsilon}{d\epsilon} = x \tag{A.72}$$

$$\implies x(\epsilon) = x(0)e^{\epsilon} \tag{A.73}$$

But we simply write the finite scaling as $x' = \lambda x$, and we now deduce $\lambda = e^{\epsilon}$. 'Exponentiated' indeed.

Now consider the ϵ - variation of the field $\phi(x)$,

$$\phi(x)(\epsilon) = \phi(x) - i\epsilon D\phi(x) \tag{A.74}$$

We want to know the details of $\tilde{\Delta}$. So, put $x = 0$. Then $D\phi(0) = \tilde{\Delta}\phi(0)$ by definition. And also, let's write $\tilde{\Delta} = -i\Delta$. This gives us the following infinitesimal flow,

$$\phi(0)(\epsilon) = \phi(0) - \epsilon\Delta\phi(0) + \epsilon^{\in} \tag{A.75}$$

Finite flow is generated as:

$$\frac{d\phi(0)(\epsilon)}{d\epsilon} = -\Delta\phi(0) \tag{A.76}$$

$$\phi_{\epsilon}(0) \equiv \phi'(0) = (e^{-\epsilon})^{\Delta}\,\phi(0)$$

$$\phi'(0) = \lambda^{-\Delta}\phi(0) \tag{A.77}$$

So, indeed, the connection is rightly captured. In fact, even for Lorentz transformations in 2 dimensions, we sort of get a similar scale-like factor, but for spin. ~ Spin dimension. We will comment on that in the 2D CFT section. We considered the fields at $\phi(0)$ and looked at its exponentiation under scaling. What would be the result if we studied fields at arbitrary positions? This, indeed, takes us to the next section.

Appendix B

Quantum Symmetries in QFT

The main result of using Path Integral formulation to quantize a classical field theory is that the correlation functions can be written as:

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle = \frac{1}{Z} \int [d\phi] \phi(x_1) \ldots \phi(x_n) e^{-S[\phi]} \tag{B.1}$$

in the Euclidean or Imaginary time setting. Under symmetry transformations (invariant action), we have

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle = \langle \mathcal{F} \left(\phi(x_1) \right) \ldots \mathcal{F} \left(\phi(x_n) \right) \rangle \tag{B.2}$$

In reaching the above result, we have changed the dummy integration variables and used the form A.20 along with the invariance of action. But also importantly, we have assumed that the measure respects the symmetry $[d\phi'] \sim [d\phi]$ - or that the Jacobian of such a change of variables $\phi' \rightarrow \phi$ is trivial (not depending on ϕ). Equation B.2 can be applied to translations, Lorentz rotations and scaling as follows:

Translation: $x' = x + a$

$$\langle \phi(x_1') \ldots \phi(x_n') \rangle = \langle \phi(x_1) \ldots \phi(x_n) \rangle \tag{B.3}$$

Lorentz:(Scalar Fields) $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$

$$\langle \phi(\Lambda^{\mu} x_1^{\nu}) \ldots \phi(\Lambda^{\mu}_{\nu} x_n^{\mu}) \rangle = \langle \phi(x_1) \ldots \phi(x_n) \rangle \tag{B.4}$$

Scaling: $x'^{\mu} = \lambda x^{\mu}$

$$\langle \phi(\lambda x_1) \ldots \phi(\lambda x_n) \rangle = \lambda^{-\Delta_1} \ldots \lambda^{-\Delta_n} \langle \phi(x_1) \ldots \phi(x_n) \rangle \tag{B.5}$$

B.1 Ward Identity

Let X denote the product of n fields, whose correlator is given by,

$$
\begin{aligned}
\langle X \rangle &= \frac{1}{Z}\int [d\phi] X e^{-S[\phi]} &\text{(B.6)}\\
&= \frac{1}{Z}\int [d\phi'] X' e^{-S[\phi']}\\
&= \frac{1}{Z}\int [d\phi] (X + \delta X) e^{-S[\phi] - \int dx \partial_\mu j^\mu_a \omega_a(x)}\\
&= \frac{1}{Z}\int [d\phi] X e^{-S[\phi]} + \frac{1}{Z}\int [d\phi] \delta X e^{-S[\phi]} - \frac{1}{Z}\int [d\phi] \left\{ \int dx \partial_\mu j^\mu_a \omega_a(x) + O(\omega_a^2) \right\}
\end{aligned}
$$

This gives,

$$
\langle \delta X \rangle = \int dx \partial_\mu \langle j^\mu_a(x) X \rangle \omega_a(x) \tag{B.7}
$$

But through definition A.21 we also have,

$$
\begin{aligned}
\delta X &= -i \sum_{i=1}^{n} \left(\phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n) \right) \omega_a(x_i) + O(\omega_a^2)\\
&= -i \int_{\text{all space}} dx\ \omega_a(x) \sum_{i=1}^{n} \left\{ \phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n) \right\} \delta(x - x_i) &\text{(B.8)}
\end{aligned}
$$

Using this in equation B.7, we get the Ward Identity for j^μ_a:

$$
\frac{\partial}{\partial x^\mu} \langle j^\mu_a(x) \phi(x_1) \dots \phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n) \rangle \tag{B.9}
$$

The form of j^μ_a may be modified from the canonical definition without changing the ward identity by adding a divergenceless (identically without using the EOM) term. Basically, this gives us the idea of how the classical N oethers theorem translates into the quantum theory within the correlation functions.

Integrating the above identity overall space, including all x_i makes the LHS vanish, giving,

$$\delta_\omega \langle \phi(x_1)\dots\phi(x_n)\rangle \equiv -i\omega_a \sum_{i=1}^{n} \langle \phi(x_1)\dots G_a\phi(x_i)\dots\phi(x_n)\rangle = 0$$

Variation of correlator under infinitesimal transformations vanishes! This is basically the infinitesimal version of equation B.2 (by just noting that $\mathcal{F}\left(\phi(x_i) \equiv \phi'(x_i')\right)$).

Ward Identity allows us to identify the conserved charge as the generator of the symmetry transformation on the Hilbert Space of Quantum States!

First, let's look at the calculation, and then we will recall the connection between Noether charge and quantum symmetries.

Define the following,

- $Q_a = \int d^{d-1} j_a^0(x)$,
- $Y = \phi(x_2)\dots\phi(x_n)$,
- $t = x_i^0 \neq x_2^0 \neq \dots \neq x_n^0$.

Integrate the Ward Identity in a *thin pill box* -

$$\begin{aligned} t_- < t < t_+ &\text{ such that } x_2, \dots x_n \text{are excluded} \\ -\infty < \bar{x} < \infty \end{aligned}$$

Start with L.H.S -

$$\begin{aligned} \int dx \partial_\mu \langle j_a^\mu(x)\phi(x_)Y\rangle &= \int dx \partial_0 \langle j_a^0(x)\phi(x_1)Y\rangle + \int dx \partial_i \langle j_a^i(x)\phi(x_1)Y\rangle \\ &= \int dt \partial_t \langle Q_a(t)\phi(x_1)Y\rangle + \int dt \int \cancel{d\bar{x}\partial_i \langle j_a^i(x)\phi(x_1)Y\rangle} \\ &= \langle Q_a^+\phi(x_1)Y\rangle - \langle Q_a^-\phi(x_1)Y\rangle \end{aligned} \tag{B.10}$$

And the r.h.s reads - since the space includes only $x = x_1$,

$$R.H.S = -i \langle G_a \phi(x_1) Y \rangle$$

and in the limit $t_- \to t_+$ of the operator formalism (where correlation functions are the time ordered expectation value of fields), we have

$$\langle 0| \left[Q_a, \phi(x_1) \right] Y |0\rangle = -i \langle 0| G_a \phi(x_1) |0\rangle \tag{B.11}$$

and since Y is arbitrary, we have,

$$\left[Q_a, \phi \right] = -i G_a \phi. \tag{B.12}$$