# INTRODUCTION TO MANIFOLDS

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Abstract. These are notes taken as an effort to consolidate the material by the author for the course *Introduction to Manifolds* at NISER taught by Dr. Chitrabhanu Chowdhury during the even semester 2024-2025. It's a combination of lecture scribes, detail filling and personal notes connecting the material.

## Contents

# 1. Manifolds

## 1.1. Topological Manifolds.

**Definition 1.1.** A topological manifold $X$ of dimension $n$ is a topological space such that,

- $X$ is Hausdorff and second countable.
- (Locally Euclidean) For any $p \in X$, there is a neighorhood $U$ such that there is a homoemorphism $\phi : U \to V$, where $V \subseteq \mathbb{R}^n$ is an open subset.

$(U, \phi)$ is called a **chart** of $X$ (containing $p$). It is called a chart cenetered at $p$ if $\phi(p) = 0$. One can naturally wonder if the dimension of a manifold is unique. Indeed, we ask:

*Question* 1.2. Can a 2-dimensional manifold be a 3-dimensional manifold (and vice-versa)?

The answer is no due to the following theorem on Euclidean spaces.

**Theorem 1.3** (Invariance of Domain)**.** *If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open sets and $\phi : U \to V$ is a homeomorphism, then $n = m$.*

*Proof.* One can show this for $n = 1$ by using a simple connectedness argument. The general proof however relies on the machinery of homology groups. □

Local Euclideanness of manifolds then immediately gives the following.

**Corollary 1.4.** *If $X$ is a topological manifold of dimensions $n$, it can't be a topological manifold of any other dimension.*

One can still talk about different dimensions by dealing with connected components where the dimension is a locally constant function; but we won't be dealing with such cases here.

### 1.1.1. *Examples.* 

Using the fact that $\mathbb{R}^n$ is a second countable and Hausdorff space, we have the following elementary examples of manifolds.

**Example 1.5.** $\mathbb{R}^n$ is a topological manifold of dimension $n$. $(\mathbb{R}^n, Id)$ covers $\mathbb{R}^n$ as a single chart.

**Example 1.6.** $U \subseteq \mathbb{R}^n$ is a topological manifold of dimension $n$.

**Example 1.7** (A cusp)**.** $K = \{(x, y) \,|y^2 = x^3\}$ is a topological manifold of dimension 1. Take the projection onto the y axis. That is, consider $\phi : K \to \mathbb{R}$ with $\phi(x, y) = y$ which has the inverse, $\phi^{-1} : \mathbb{R} \to K$ with $\phi^{-1}(y) = \left(y^{\frac{2}{3}}, y\right)$. $\phi$ is then a homeomorphism and $((K, \phi))$ is a chart. Any $X$ which is homeomorphic to $K$ is also thus a 1 dimensional topological manifold.

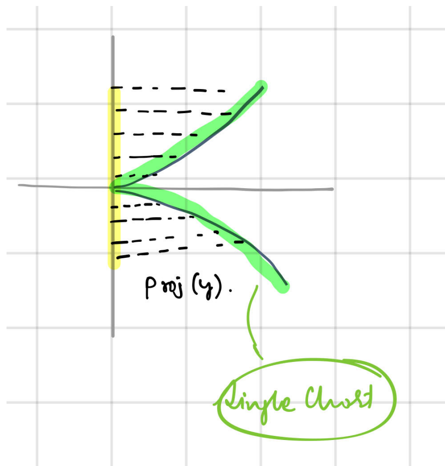



FIGURE 1. A 1 dimensional topological manifold: Cusp.

**Example 1.8** (Cone).

$$X = \left\{(x, y, z) \in \mathbb{R}^3 | z = x^2 + y^2, z \geq 0\right\}$$

Take a projection onto the xy plane, $\phi : X \to \mathbb{R}^2, \phi(x, y, z) = (x, y)$. This gives the homeomorphism to $\mathbb{R}^2$. Thus, a cone is a 2d topological manifold covered by $(\phi, X)$.

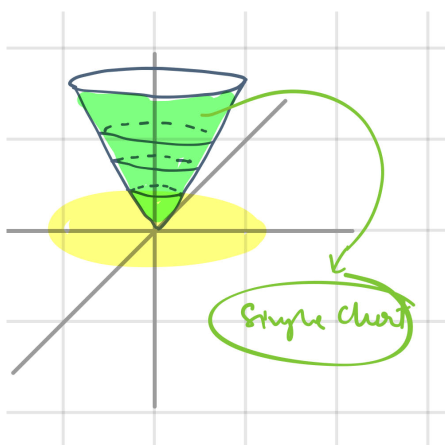



FIGURE 2. Cone as a 2 dimensional manifold.

**Example 1.9** (Circle).

$$S^1 = \left\{(x, y) \, | x^2 + y^2 = 1\right\}$$

Although not homeomorphic to $\mathbb{R}$ it is locallyi Euclidean. To see this, consider the folowing charts:

- $U_1 = \{(x, y) \in S^1 \mid y > 0\} = \{(x, \sqrt{1 - x^2} \mid x \in (-1, 1))\}$which is homeomorphic to $(-1, 1) \subseteq \mathbb{R}$, through $\phi_1 : U_1 \to \mathbb{R}, \phi_1(x, y) = x$.
- $U_2 = \{(x, y) \in S^1 \mid y < 0\} = \{(x, -\sqrt{1 - x^2} \mid x \in (-1, 1))\}$which is homeomorphic to $(-1, 1) \subseteq \mathbb{R}$, through $\phi_2 : U_2 \to \mathbb{R}, \phi_2(x, y) = x$.
- Similarly $U_3$ and $U_4$ projecting them onto y axis: $(-1, 1)$.

$S^1$ is thus a topological 1-manifold.

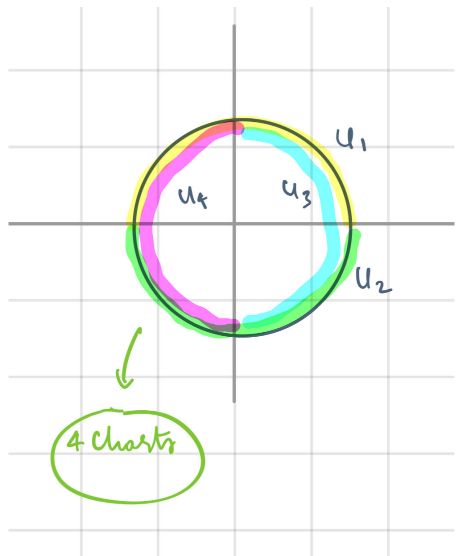



FIGURE 3

*Exercise* 1.10. Similarly, $S^n$ is a topological manifold of dim. $n$.

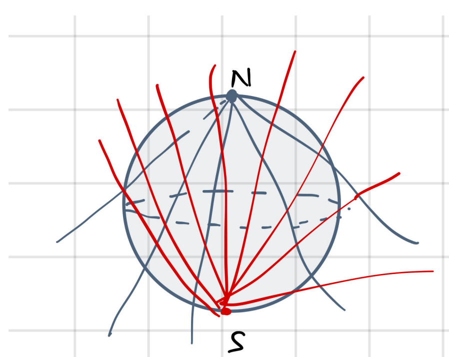
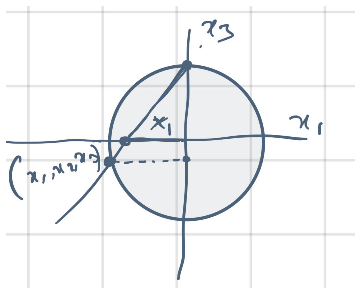
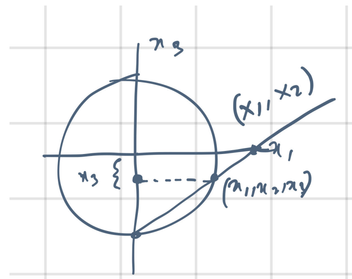



FIGURE 4

*Proof.* We can show this by using stereographic projection of the $n$-sphere. Let $N = (0, \dots, 0, 1), S = (0, \dots, 0, -1) \in S^n$. Define $U = S^n \setminus N, V = S^n \setminus S$ and




(A) Projection near the North Pole.




(B) Projection near the South Pole.

$$
\begin{aligned}
&\phi : U \to \mathbb{R}^n \\
&\qquad x \mapsto \phi\left(x_1, \ldots, x_{n+1}\right) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\right).
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi : V \to \mathbb{R}^n \\
&\qquad x \mapsto \psi\left(x_1, \ldots, x_{n+1}\right) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}}\right).
\end{aligned}
$$

The charts $(U, \phi), (V, \psi)$ cover $S^n$ as an n-dim. manifold. □

1.1.2. *Non Examples.*

**Example 1.11.** Any discrete space (a set with discrete topology) which is uncountable is not a manifold, because it's not second countable. And thus, $\mathbb{R} \times D$ where $D$ is a discrete uncountable space is also not a manifold.

*Exercise* 1.12 (Real line with double origin). $X = \mathbb{R} \times \{0, 1\} / \sim$ with $(x, 0) \sim (x, 1)$ if $x \neq 0$.

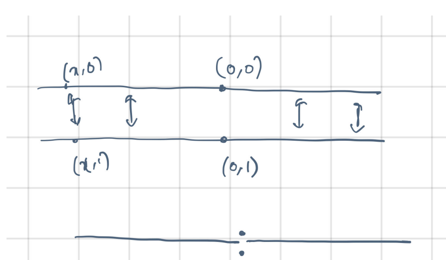



FIGURE 6

*Proof.* We will prove this in two ways, once with and once without envoking the quotient topology. We will come back to this proof soon. □

**Example 1.13.** $Y = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is not localle Euclidean at $(0, 0)$. Consider a neighborhood $U \subseteq Y$ of 0. Assume $U$ is connected (or take a connected component containing 0). Suppose it is locally Euclidean at 0. Then, $\exists$

$$\phi : U \to (a, b) \subset \mathbb{R}$$

a homeomorphism with say $h(0) = c$. It's restriction $h : U \setminus \{(0, 0)\} \to (a, c) \cup (c, b)$ is also a homeomorphism. This is a contradiction since, $U \setminus \{(0, 0)\}$ has 4 connected components while $(a, c) \cup (c, b)$ has only two.

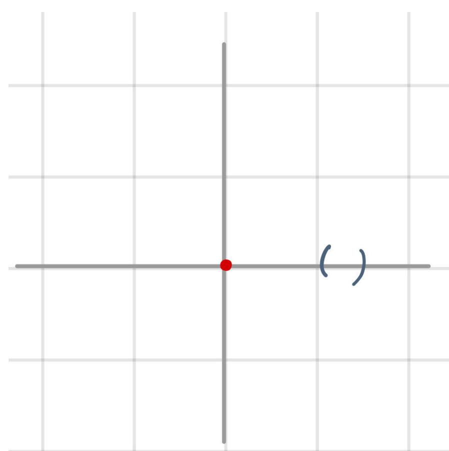


FIGURE 7

**Example 1.14.** $Z = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ is not a topological manifold. This too follows from a connectedness argument. If $Z$ were to be locally Euclidean around 0, there is a homeomorphism of the neighborhood around 0 (simply a small open ball in $\mathbb{R}^3$ intersected with $Z$) to the open disc $D^2$ in $\mathbb{R}^2$. Restricting this to $Z \setminus 0$ gives a disparity in the conceted components (two in $Z \setminus 0$ and none in $D^2$).

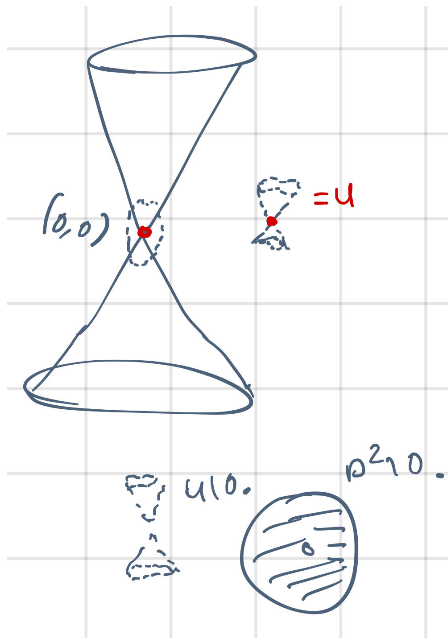



FIGURE 8

**Example 1.15.** $X = S^2 \cup (D^2 \times \{0\})$ is not a topological manifold. This is basically a sphere with a 2d disc attached. The problem is at $S^1 \times \{0\} \subset X$. One needs to envoke local homology groups to prove this statement.

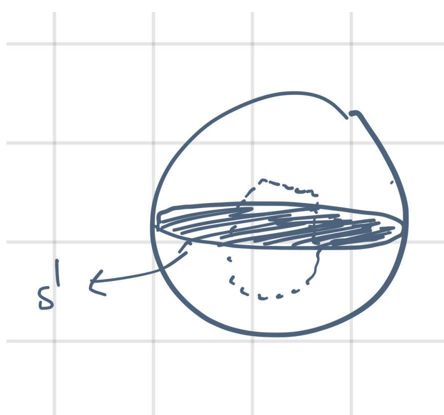



FIGURE 9

1.1.3. *Properites of a topological manifold.*

(1) Locally Compact.
(2) Locally path connected.
(3) Connected components are path component.

(4) There are Countably many connceted components.

These are one of the herculean tasks. We will come back to prove these.

1.2. **Smooth Manifolds.** Let $M$ be a topological manifold, and $(U, \phi)$ and $(V, \psi)$ be two charts of $M$. The two charts are called (smoothly or $C^\infty$) compatible if the *transition functions*

$$\begin{aligned} \psi \circ \phi^{-1} : & \phi(U \cap V) \to \psi(U \cap V) \\ \phi \circ \psi^{-1} : & \psi(U \cap V) \to \phi(U \cap V). \end{aligned}$$

are both smooth. The idea is to develop the notion of smoothness of a manifold, so that it suffices to check smoothness with a single chart around a given point.

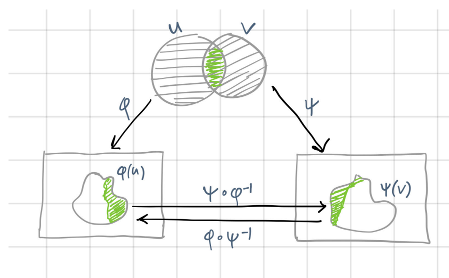



FIGURE 10

**Definition 1.16** (Smooth Atlas)**.** Let $M$ be a topological manifold. A smooth atlas $\mathcal{A}$ on $M$ is a set of charts $\{(U_i, \phi_i) \mid i \in I\}$ such that,

(1) $(U_i, \phi_i)$ and $(U_j, \phi_j)$ are compatible for all $i, j \in I$.
(2) $M = \bigcup_{i \in I} U_i$.

**Definition 1.17** (Smooth manifold)**.** A smooth structure on a topological manifold $M$ is a maximal smooth atlas $\mathcal{A}$ on $M$. A smooth manifold is a pair $(M, \mathcal{A})$ is a maximal smooth atlas on $M$.

*Remark* 1.18. A smooth atlas is called *maximal* if it is not properly contained in any other smooth atlas. This will be useful to deal with different structures given by different smooth atlasses. Any smooth atlas is infact contained in a unique maximal atlas. Looking for maximal atlas is almost hopeless. We have the following neat result which makes things easier. Further, in two and three dimensions all smooth structures are diffeomorphic!

**Proposition 1.19.** *Let $M$ be a topological manifold.*

(1) *Any smooth atlas of $M$ is contained in a unique maximal smooth atlas.*
(2) *Two smooth atlases are contained in the same maximal atlas if and only if their union is a smooth atlas.*

Thus, finding one smooth atlas for a manifold suffices to give a unique smooth structure to a topological manifold. However, it is a difficult and active question to distinguish two smooth strucutres or to find all the smooth strucutres on a manifolds.

## 2. Submanifolds

**Theorem 2.1** (Inverse Function Theorem for manifolds). *Let $F : M \to N$ be a smooth map,*

(1) *If $F_{*p} : T_pM \to T_{F(p)}N$ is an isomorphism, then $\exists$ a neighborhood $V$ of $p$ and a neighborhood $U \in F(p)$ s.t. $F(V) = U$ and*
$$F : V \to U \text{ is a diffeomorphism.}$$
(2) *If $F_{*p} : T_pM \to T_{F(p)}N$ is an isomorphism for all $p \in M$, then $F$ is a local diffeomorphism.*
(3) *If $F$ is a bijection and $F_* : T_pM \to T_{F(p)}N$ is an isomorphism for all $p \in M$, then $F$ is a diffeomorphism.*

*Proof.* Suppose $n = dim(M) = dim(N)$. Let $(V, \psi)$ be a chart centered at $F(p)$. And $(U, \phi)$ be a chart centered at $p$. Then,

$$G = \psi \circ F \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^n, G(0) = 0$$

has an invertible jacobian matrix $J_G(0)$ because $F_{*p}$ is injective. So, by Inverse function theorem on Euclidean spaces, $\exists U_0' \subseteq \phi(U)$ and $0 \in U_0'$ s.t. $V_0' = G(U_0')$ is also open in $\psi(V)$ and $G : U_0' \to V_0'$ is a diffeomorphism. Then,

$$F = \psi^{-1} \circ G \circ \phi : U_0 := \phi^{-1}(U_0') \to V_0 := \phi^{-1}(V_0') \text{ is also a diffeomorphism.}$$

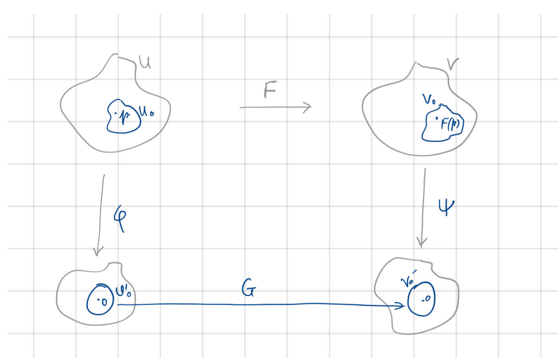



Figure 11

□

## 2.1. Regular Submanifolds.

**Definition 2.2.** Let $N$ be a smooth n-manifold and $M \subseteq N$. Then $M$ is called a regular submanifold of $dim.k$ of $N$ if for any $p \in M$, $\exists$ a chart $(U, \phi)$ of $N$ s.t.

$$U \cap M = \{\phi^{-1}(x_1, \ldots, x_n)\, x \in \phi(U) \mid x_{k+1} = \ldots = x_n = 0\}$$

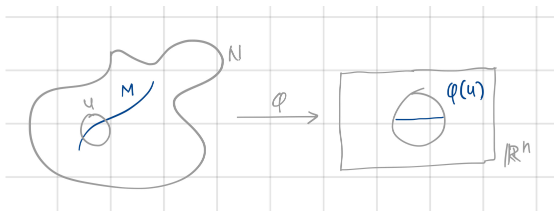



FIGURE 12. Regular Submanifold $M$ of $N$. This is basically a prototype of $\mathbb{R}^k$ living inside $\mathbb{R}^n$.

**Remark.** $n - k = dim(N) - dim(M)$ is called the co-dimension of the submanifold $M$. We can cover $M$ by charts satisfying the above property. Let,

$$\begin{aligned} \phi_M : U \cap M \to \mathbb{R}^k \\ \phi_M(q) = (\phi_1(q), \ldots, \phi_k(q)) \end{aligned} \tag{2.1}$$

$\phi_M : U \cap M \to \phi_M(U \cap M) \subseteq \mathbb{R}^k$ is continous, bijection and open and thus a homeomorphism (using the projection map which is open and continous).

$(U, \phi)$ is called an adapted chart of $M$. Now we check for smoothness. Let $(U, \phi)$, $(V, \psi)$ be two adapted charts of $M$,

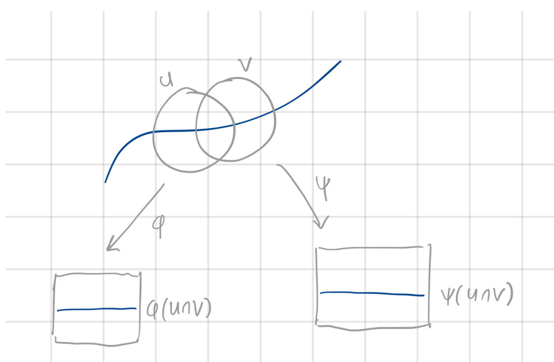



FIGURE 13

We want to show $\psi_M \circ \phi_M^{-1} : \phi(U \cap V capM) \to \psi(U \cap V \cap M)$ is smooth. $\psi_M \circ \phi_M^{-1}$ is given by,

$$\left(\psi_1\left(\phi^{-1}\left(x_1,\ldots,x_k,0,\ldots,0\right)\right),\ldots,\psi_k\left(\phi^{-1}\left(x_1,\ldots,x_k,0,\ldots,0\right)\right)\right)$$

and is indeed smooth. So we have,

**Proposition 2.3.** *A regular submanifold has a unique smooth strucutre where the adapted charts form an atlas.*

**Remark.** Any manifold can be realized as a regular submanifold of some $\mathbb{R}^d$.

**Examples.**

1. $M$ is an n-manifold, $U \subseteq M$ is an open subset, then $U$ is a regular submanifold of *dim.n*. Moreover, if $N \subseteq M$ is a regu;lar submanifold of *dim.k* then $N$ is an open subset of $M$. So submanifolds of co-dimension 0 are just open ssubsets.

2. $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$ is a submanifold of dim. k.

3. $(-1,1) \times \{0\}$ is a submanifold of $\mathbb{R}^2$ of dim. 1. The adapted chart is $((-1,1) \times \mathbb{R}, Id)$.

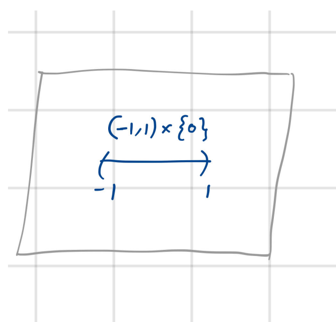



FIGURE 14

4. $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 0; x,y \geq 0\}$ is not a regular sub manifold of $\mathbb{R}^2$.

5. $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2; z \geq 0\}$ is not a regular submanifold of $\mathbb{R}^3$.

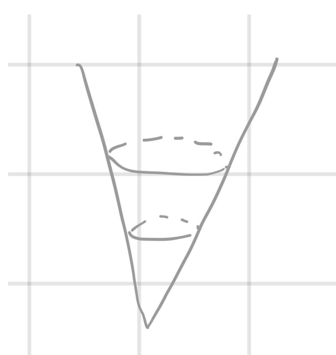


FIGURE 15

**Definition 2.4.** If $F : M \to N$ is a smooth map between smooth manifolds,

(1) A point $p \in M$ is called a criticial point if $F_{*p} : T_pM \to T_{F(p)}N$ is not surjective.

(2) A point $q \in N$ is called a critical value if $F^{-1}(q)$ has a critical point, otherwise $q$ is called a regular value.

**Remark.** $q \in N$ is a regular value if either $F^{-1}(q)$ is empty or $\forall p \in F^{-1}(q)$, $F_{*p}$ is surjective. Note that if dim. $M >$ dim. $N$, then all points of $N$ are regular. Sard's theorem asserts that the set of critical values is a null set (lesbegue measure 0). Note also that

$$F_{*p} \text{ is injective } \iff m < n, rk\left(J_F\right) \text{ is maximal (m).}$$

$$F_{*p} \text{ is surjective } \iff m > n, rk\left(J_F\right) \text{ is maximal (n).}$$

**Examples.**

1. Let $F : M \to \mathbb{R}$ be a smooth map, then note that $p \in M$ is a critical point of $F$ only if $F_* = 0$, i.e. if $(U, \phi)$ is a chart around $p$, then

$$\frac{\partial F \circ \phi^{-1}}{\partial x_i}(\phi(p)) = 0, \ \forall i = 1, \dots, n.$$

2. $F : M_n(\mathbb{R}) \to SM_n\left(\mathbb{R}\right), F(A) = AA^t$. Note that $F^{-1}(I) = O\left(n\right)$ the set of all $n \times n$ orthogonal matrices.

Suppose $A \in O(n)$, then what is the rank of $F_* : T_AM_n(\mathbb{R}) \to T_ISM_n(\mathbb{R})$?

Let $\gamma(t) = A + tB, \gamma : \mathbb{R} \to M_n(\mathbb{R})$ be a curve starting passing through $A$ with $\gamma'(0) = B \in T_AM_n(\mathbb{R})$. Then,

$$\begin{aligned} F(\gamma(t)) &= AA^t + t^2BB^t + t\left(AB^t + BA^t\right) \\ &= F_*\left(\gamma'(0)\right) = \left(F \circ \gamma\right)'(0) = AB^t + BA^t \in T_ISM_N(\mathbb{R}). \end{aligned}$$

Now, for any $C \in SM_n(\mathbb{R})$, let $B = \frac{1}{2}CA$. Then $F_*(B) = C$. So, $F_*$ is surjective for all $A \in F^{-1}(I)$. So $I$ is a regular value of $F$.

3. Consider $f = det : M_n(\mathbb{R}) \to \mathbb{R}$. Then $SL(n, \mathbb{R}) = f^{-1}(1)$. To show 1 is a a regular value of $f$ we write,

$$\begin{aligned} det(A) &= \sum_{j=1}^{n}(-1)^{i+j}a_{ij}det(M_{ij}) \\ \frac{\partial f(A)}{\partial a_{ij}} &= (-1)^{i+j}\, det(M_{ij}). \end{aligned} \tag{2.2}$$

If $A$ is a critical point of $f$, then $det(M_{ij}) = 0 \ \forall i, j \implies det(A) = 0$. So indeed 1 is a regular value of $f$.

Regular submanifolds are precisely characterized by these regular values!

**Theorem 2.5** (Regular Level Set/Regular Value)**.** *If $F : N^{(n)} \to M^{(m)}$ is a smooth map and $c \in M$ is a regular value of $F$, then $F^{-1}(c)$ is a regular submanifold of $N$ of co-dimension $m$.*

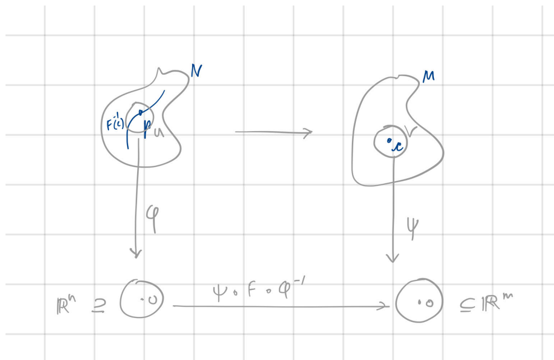



FIGURE 16

*Proof.* For $p \in F^{-1}(c)$, choose a chart $(U, \phi)$ centered at $p$ and a chart $(V, \psi) \ni c$ and $\psi(c) = 0$. Assume $F(U) \subseteq V$. So, $\psi \circ F \circ \phi^{-1}(0) = 0$. Further, $F_* : T_pN \to T_cM$ is surjective, so the rank of $J_{\psi \circ F \phi^{-1}}(0)$ is $m$. This means $J_{\psi F \phi^{-1}}$ can be written as,

$$J_{\psi F \phi^{-1}} = \left[A|B\right] \tag{2.3}$$

where $A$ is an $m \times (n-m)$ matrix and $B$ is an invertible $m \times m$ matrix. Consider now,

$$\begin{aligned} \theta &: U \to \mathbb{R}^n \\ \theta(q) &= \left(\phi_1(q), \ldots, \phi_{n-m}(q), (\psi F)_1(q), \ldots, (\psi F)_m(q)\right). \end{aligned} \tag{2.4}$$

then

$$\begin{aligned} \theta \circ \phi^{-1} &: \phi(U) \to \mathbb{R}^n \\ \theta \circ \phi^{-1}(x) &= \left(x_1, \ldots, x_{n-m}, \psi \circ F \circ \phi^{-1}\right) \end{aligned} \tag{2.5}$$

has the following jacobian,

$$J_{\theta \circ \phi^{-1}}(0) = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \tag{2.6}$$

and is invertible since $B$ is invertible.

So, $\exists W \subseteq \phi(U)$ open containing 0 s.t. $\theta \circ \phi^{-1}(W)$ is open, and

$$\theta \circ \phi^{-1} : W \to \theta \circ \phi^{-1}(W) \text{ is a diffeomorphism.}$$

Consider, $U_0 = \phi^{-1}(W) \subset U$. Then, $(U_0, \theta)$ is an adapted chart for $F^{-1}(c)$ around $p$. Since, For $q \in U_0 \cap F^{-1}(c)$,

$$\theta(q) = (x_1, \ldots, x_{n-m}, 0, \ldots, 0).$$

□

We should check this for $S^n \subset \mathbb{R}^{n+1}$.

**Theorem 2.6** (Transversality)**.** *If $F : M \to N$ is a smooth map and $P \subseteq N$ is a submanifold. Then $F$ is called transverse to $P$ is $\forall c \in P$ and all $q \in F^{-1}(c)$,*

$$F_* T_q M + T_c P = T_c N$$

*If this holds, then $F^{-1}(P)$ is a submanifold of $M$.*

We will now through some results try to establish when an image of a smooth map is a submanfild.

**Theorem 2.7** (Constant Rank - Euclidean Case)**.** *Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ a smooth map s.t. $rk J_f(x) = k \ \forall x \in U$. Then for any $p \in U \exists$ an open set $U_0 \subseteq U$ containing $p$ and a neighborhood $V_0$ of $f(p)$ in $\mathbb{R}^m$, and diffeomorphisms $\phi : U_0 \to \phi(U_0)$ and $\psi : V_0 \to \psi(V_0)$ s.t. $f(U_0) \subset V_0$ and $\psi \circ f \circ \phi^{-1}(x) = (x_1, \ldots, x_k, 0, \ldots, 0)$.*

This says, $U$ is a regular submanifold of $\mathbb{R}^n$.

*Proof.* Let $p = 0, f(p) = 0$ (WLOG). We have,

$$J_f\,(0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.7}$$

WLOG, assume $A$ is an invertible $k \times k$ matrix.

Then define,

$$\begin{aligned} &\phi : U \to \mathbb{R}^n \\ &\phi(x) = (f_1(x), \ldots, f_k(x), x_{k+1}, \ldots, x_n) \end{aligned} \tag{2.8}$$

which has a jacobian,

$$J_\phi(0) = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \tag{2.9}$$

and is invertible because $A$ is invertible. Then $\exists$ a connected neighborhood $U_0$ of 0 s.t. $\phi : U_0 \to \phi(U_0)$ is a diffeomorphism. Denote $U' = \phi(U_0)$.

For $x \in \mathbb{R}^n, x = (u, v)$ where $u = (x_1, \ldots, x_k)\,, v = (x_{k+1}, \ldots, x_n)$. We can write,

$$\begin{aligned} &f \circ \phi^{-1} : U' \to \mathbb{R}^n \\ &f \circ \phi^{-1}(u, v) = (u, h(u, v)) \end{aligned} \tag{2.10}$$

where,

$$\begin{aligned} &h : U' \to \mathbb{R}^{m-k} \\ &h(u,v) = \left(f_{k+1} \circ \phi^{-1}(u,v), \ldots, f_m \circ \phi^{-1}(u,v)\right) \end{aligned} \tag{2.11}$$

Then, $J_{f\circ\phi^{-1}}(u,v) = J_f\left(\phi^{-1}(u,v)\right) J_{\phi^{-1}}(u,v)$ has rank $k$, since $J_{\phi^{-1}}$ has full rank $n$. But we have,

$$J_{f\circ\phi^{-1}}(u,v) = \begin{pmatrix} I_k & 0 \\ C'(u,v) & D'(u,v) \end{pmatrix} \tag{2.12}$$

which means $D'(u,v) = 0$ as $J$ doesn't have $n \times n$ non-singular sub matrix $(k \neq n)$. We then have $h(u,v) = h(u)$ giving $f \circ \phi^{-1}(u,v) = (u, h(u))$.

Now, for $y \in \mathbb{R}^n, y = (u,w)$ where $u = (y_1, \ldots, y_k), w = (y_{k+1}, \ldots, y_n)$ define

$$\psi : V' \times \mathbb{R}^{m-k} \to \mathbb{R}^m \tag{2.13}$$

$$\psi(u,w) = (u, w - h(u)) \tag{2.14}$$

where, $V' = \{u \in \mathbb{R}^k \mid (u,0) \in U'\}$. $\psi$ is a diffeomorphism onto its image! And we have,

$$\psi \circ f \circ \phi^{-1}(u,v) = \psi(u, h(u)) = (u, h(u) - h(u)) = (u, 0).$$

$\square$

**Theorem 2.8** (Constant Rank - Manifold Case). *Let $F : N^{(n)} \to M^{(m)}$ be a smooth map. Suppose $p \in N$ has a neighborhood $U$ s.t.*

$$F_* : T_qN \to T_F(q)M \text{ has a rank } k \ \forall q \in U,$$

*then there are charts $(U_0, \phi)$ around $p$ in $N$ and $(V_0, \psi)$ around $F(p)$ in $M$, s.t. $F(U_0) \subseteq V_0$ and $\psi \circ F \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$.*

*Proof.* Let $(V_1, \theta)$ be a chart of $M$ centered at $F(p)$, and $(U_1, \zeta)$ be a chart in $N$ centered at $p$ s.t. $U_1 \subseteq U$ and $F(U_1) \subseteq V_1$. Apply the last theorem to the following map,

$$f = \theta \circ F \circ \zeta^{-1} : \zeta(U_1) \subseteq \mathbb{R}^n \to \theta(V_1) \subseteq \mathbb{R}^m.$$

$\square$

**Corollary 2.9** (Constant Rank Level Set). *Suppose $F : M \to N$ is a smooth map, and $c \in M$ is s.t. $\exists U \subseteq N$ open and $F^{-1}(c) \subseteq U$ for which,*

$$F_* : T_qN \to T_F(q)M \text{ has a rank } k \ \forall q \in U.$$

*Then, $F^{-1}(c)$ is a regular submanifold of codimension $k$.*

*Proof.* For $p \in F^{-1}(c)$, by constant rank theorem, there are charts $(U, \phi)$ containing $p$ and $(V, \phi)$ containing $F(p) = c$ and $\psi(c) = 0$ s.t.

$$\psi \circ F \circ \phi^{-1}(x) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

$$\begin{aligned} \phi(U \cap F^{-1}(c)) &= \phi \circ F^{-1}(c) \\ &= \phi \circ F^{-1} \circ \psi^{-1}(0) \\ &= \left(\psi \circ F \circ \phi^{-1}\right)^{-1}(0) \\ &= \{x \in \phi(U) \mid x_1 = \cdots = x_n = 0\} \end{aligned} \tag{2.15}$$

So, $(U, \phi)$ is an adapted chart of $F^{-1}(c)$. ☐

2.2. **Immersions and Submersions.** Note that the rank of $F_*$. where $F : M \to N$ is a smooth map can increase in a neighborhood but won't decrease, because the *det* remains non-zero in a neighborhood of $p$, if $det|_p \neq 0$.

**Definition 2.10.** Let $F : M \to N$ be a smooth map.

(1) F is called an immersion at $p \in M$ if $F_* : T_pM \to T_{F(p)}N$ is injective. F is called an immersion if it is an immersion at all points of $M$.
(2) F is called a submersion at $p \in M$ if $F_* : T_pM \to T_{F(p)}N$ is surjective. And $F$ is called a submersion if it is a submersion at all points of $M$.

**Corollary 2.11.** *Let $F : M^{(m)} \to N^{(n)}$ be a smooth map.*

(1) *If $F$ is an immersion at $p$, then there are charts $(U, \phi)$ centered at $p$ and $(V, \psi)$ centered at $F(p)$ s.t. $F(U) \subseteq V$ and*

$$\psi \circ F \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

(2) *If $F$ is a submersion at $p \in M$, then there are charts $(U, \phi)$ centered at $p$ and $(V, \psi)$ centered at $F(p)$ s.t. $F(U) \subseteq V$ and*

$$\psi \circ F \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$$

This follows because the rank is maximal and is thus constant in a neighborhood of $p$ in both the cases. By constant rank theorem, the results follow immediately for $k = m$ and $k = m$.

**Remark.** The immersion is basically a prototype of an inclsuion map while a submersion is a prototype of a projection map!

**Definition 2.12.** A smooth map $F : M \to N$ is called an embedding if

(1) $F$ is an immersion.
(2) $F : M \to F(M)$ is a homeomorphism, where $F(M)$ has the subspace topology in $N$.

**Remark.** If the domain $M$ is compact then continuity and bijection $\implies$ homeomorphism (closed map), since compact subsets of a Hausdorff space are closed.

**Examples.**

1. $F : \mathbb{R} \to \mathbb{R}^2, F(t) = (t^2, t^3)$. Then $F'(t) = (2t, 3t^2), F'(0) = 0$. So $F_{*0} : T_0\mathbb{R} \to T_0\mathbb{R}^2$ has a rank 0 at $t = 0$, and thus F is not an immersion at 0. Thus, F is not an immersion.

However, $F : \mathbb{R} \to F(\mathbb{R})$ is a homeomorphism.

But, $F$ is not an embedding. Note that $F(R)$ is not a regular submanifold of $\mathbb{R}^2$, although is a topological manifold.

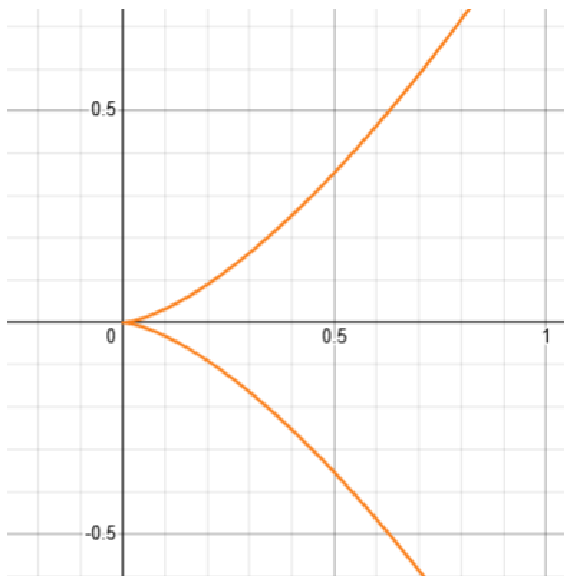



FIGURE 17

2. Consider $S = \{(x, y) \in \mathbb{R}^2 \mid sy = 0; x, y \geq 0\}$. This is an image of a smooth map $\mathbb{R} \to \mathbb{R}^2$, but that map cannot be an immersion.

For example consider,

$$f(t) = \begin{cases} \left(0, e^{t^2} - 1\right) & t \in (-\infty, 0] \\ \left(e^{t^2} - 1, 0\right) & t \in [0, \infty) \end{cases} \tag{2.16}$$

This is a smooth map, and $f(\mathbb{R}) = S$, but fails to be an immersion at $t = 0$. And $S$ is not a regular submanifold of $\mathbb{R}^2$.

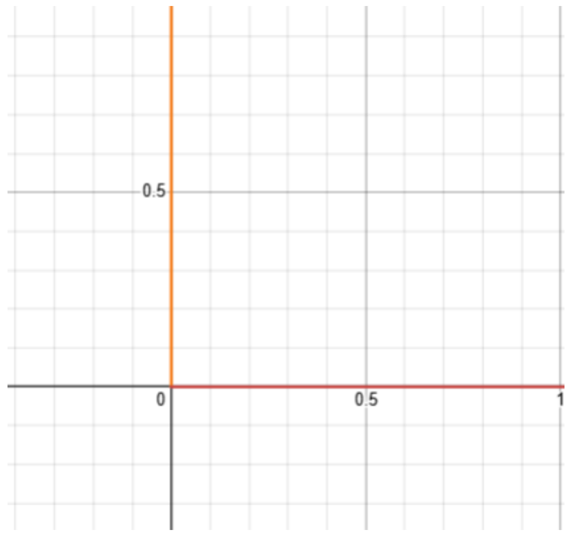



FIGURE 18

3. $g : \mathbb{R} \to \mathbb{R}^2, g(t) = (t^2 - 1, t^3 - t)$ . Then,

$$X = g(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2(x + 1)\}$$

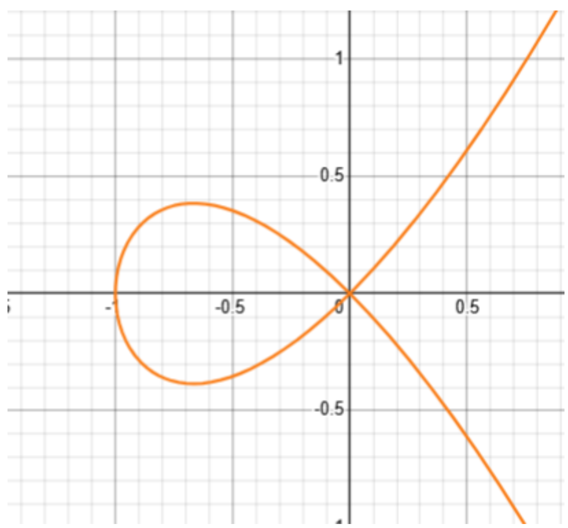



FIGURE 19

$g'(t) = (2t, 3t^2 - 1)$. So $g$ is an immersion for all $t \in \mathbb{R}$. But $g : \mathbb{R} \to X$ is not a homeomorphism. $X$ is neither a regular submanfiold of $\mathbb{R}^2$ nor a topological manifold.

4. Figure-8:

Consider,

$$f : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}^2$$
$$f(t) = (\cos t, \sin 2t) \tag{2.17}$$

$f'(t) = (-\sin t, 2\cos 2t)$. So $f$ is an immersion.

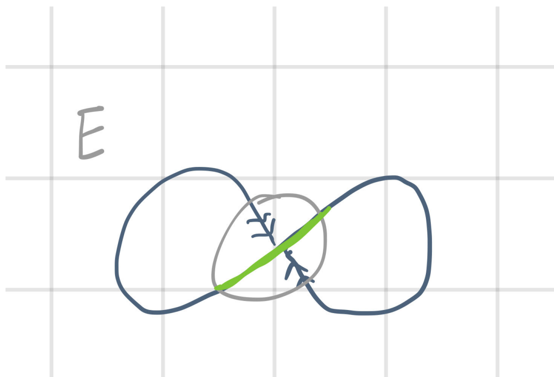



FIGURE 20

$f$ is however not homeomorphic to the image, E. And $E$ is not a regular submanifold of $\mathbb{R}^2$. The problem is at $(0, 0)$. The neighborhood is not locally Euclidean. $f\left(\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right)$ is not open in E.

**Theorem 2.13.** *If $F : M^{(m)} \to N^{(n)}$ is an embedding, then $S = F(M)$ is a regular submanifold of dim.m.*

*Proof.* For any $p \in M$, there are charts $(U, \phi)$ centered at $p$ and $(V, \psi)$ centered at $F(p)$ s.t. $F(U) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

Now, $F(U) \subseteq V \cap S$. But since $F(U)$ is open in $S$ due to the embedding, we have $V'$ open in $S$ such that $V' \cap S = F(U)$. Then we can take $(V \cap V', \psi)$ as the adapted chart of $S$ around $F(p)$. □

And infact,

**Proposition 2.14.** *If $S \subseteq M$ is a regular submanifold, then the inclusion map $i : S \to M$ is an embedding.*

We can basically find the jacobian of $i$ in local coordinates and see that the rank is $dim.M.$ at all points in S. Thus $i$ is an immersion.

The image of an embedding is called an Embedded submanifold. And the above two theorems show that it is same as the regular submanifold!

2.3. **Tangent space of a submanifold.** If $S \subseteq M$ is a submanifold, then $i : S - M$ is an embedding. So $i_* : T_pS \to T_pM$ is injective for all $p \in S$. We then identify $T_pS$ with $i_*(T_pS) = T_pS \subseteq T_pM$. So we take $T_pS$ as a subspace of $T_pM$.

**What is this subspace?**

As derivations: If $f_p \in C_p^\infty(M), v \in T_pS$. Then

$$i_*(v)(f_p) = v(f \circ i)_p) = v((f|_S)_p)$$

Recall that for any $v \in T_pS$, there is a smooth curve whose velocity vector at 0 is $v$.

**Proposition 2.15.** *If $F : M \to N$ is a smooth map, and $c \in N$ is a regular value of $F$, then $S = F^{-1}(c)$ is a regular submanifold. For $p \in S$,*

$$T_p(S) = ker\left(F_* : T_pM \to T_cN\right).$$

*Proof.* If $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve with, $\gamma\left((\epsilon, \epsilon)\right) \subseteq S, \gamma(0) = p$,

$$F \circ \gamma(t) = c \implies F_*\left(\gamma'(0)\right) = 0$$

So, $T_pS \subseteq ker\left(F_{*p}\right)$. Now, S is a manfiold of dimensions $dim(M) - dim(N)$. So,

$$dim\left(T_pS\right) = m - n = dim\left(ker\left(F_{*p}\right)\right).$$

Note that $F_{*p}$ is a surjection for all $p \in S$, since $c$ is a regular value. And thus, $dim\left(ker\left(F_{*p}\right)\right) = m - n$. So, $ker\left(F_{*p}\right) \subseteq T_pS$. □

**Examples.**

1. $S^n \subseteq \mathbb{R}^{n+1}, F : \mathbb{R}^{n+1} \to \mathbb{R}, F(x) = \|x\|^2$. Then $S^n = F^{-1}(1)$. And 1 is a regular value of $F$. Because $F'(x) = 2\left(x_1 \cdots x_{n+1}\right) = 0 \iff x = 0 \ni S^n$. So $F$ has rank 1 and is surjective for all $x \in S^n$. Then,

$$T_pS^n = ker\left(F_* : T_p\mathbb{R}^{n+1} \to T_1\mathbb{R}\right)$$

We have,

$$F_*\left(a_i\frac{\partial}{\partial x_i}|_p\right) = \left(a_i\frac{\partial F}{\partial x_i}(p)\right)\frac{d}{dt}|_1 = 0 \iff a_i\frac{\partial F}{\partial x_i}(p) = \langle a, \nabla F(p)\rangle = 2\left\langle a, p\right\rangle = 0$$

Thus, $T_pS^n = \langle p\rangle^\perp$.

2. If $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c \in \mathbb{R}$ is a regular value of $F$, and $S = F^{-1}(c)$. Then,

$$T_pS = \langle \nabla F(p)\rangle^\perp. \tag{2.18}$$

3. $SL\left(n, \mathbb{R}\right) \subseteq M_n(\mathbb{R})$ adn $det : M_n(\mathbb{R}) \to \mathbb{R}$ is a smooth function with 1 as a regular value and $SL(n, \mathbb{R}) = det^{-1}(1)$. Note that $I \in SL(n, \mathbb{R})$. What is $T_ISL(n, \mathbb{R})$?

$$T_I SL\,(n, \mathbb{R}) = ker\,(det_* : T_I M_n(\mathbb{R}) \to \mathbb{R})$$

.
Note that $T_I M_n(\mathbb{R}) = M_n(\mathbb{R})$. And consider the following smooth curve

$$\begin{aligned} \gamma &: \mathbb{R} \to M_n(\mathbb{R}) \\ \gamma(t) &= I + tA \end{aligned} \tag{2.19}$$

such that $\gamma(0) = I, \gamma'(0) = A$. So $A \in T_I M_n(\mathbb{R})$. Then,

$$\begin{aligned} det(\gamma(t)) &= det(I + tA) \\ &= t^n \chi_{-A}\left(\frac{1}{t}\right) \\ &= 1 + tr(A) + \cdots + t^n det(A) \\ det_*(\gamma(t)) &= tr(A) \end{aligned}$$

So, $T_I SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid tr(A) = 0\}$.
Note that we have used the form of characterisitc polynomial,

$$\chi_A(t) = \mid tI - A \mid = t^n - t^{n-1} tr(A) + \cdots + det(A).$$

**Remark.** The tangent space at identiy of a Lie group is called a Lie Algebra. And $SL(n, \mathbb{R})$ is a Lie Group and we identified the Lie algebra in the above example.

## 3. Vector Bundles

So far we have not linked the tangent spaces at different points. This section will discuss aspects related to that, where we can smoothly move between such vector spaces.

3.1. **Tangent Bundle.** Let M be a smooth dim. n manifold, then define,

$$TM = \bigcup_{p \in M} T_p M. \tag{3.1}$$

A point in $TM$ is a pair $(p, v)$ where $v \in T_p M$.. We will give a topology for this set, but before that let's look at the potential charts on this potential space. If $(U, \phi)$ is a chart of $M$ then any tangent vector $v \in T_p M$ for $p \in U$ can be written as,

$$v = c_i(v) \frac{\partial}{\partial x_i}\bigg|_p \tag{3.2}$$

where $\phi = (x_1, \ldots, x_n)$. Then consider $\tilde{\phi}$ a map on $TU = \bigcup_{p\in M} T_pM \subseteq TM$,

$$\begin{aligned} \tilde{\phi} : TU &\to \phi(U) \times \mathbb{R}^n \\ \left(p, c_i \frac{\partial}{\partial x_i}\bigg|_p\right) &\mapsto \left(\phi(p), (c_1, \ldots, c_n)\right). \end{aligned} \tag{3.3}$$

$\tilde{\phi}$ is a bijection, since $\phi$ is a homeomorphism. We can give a topology on $TU$ immediately by declaring $\tilde{\phi}$ is a homeomorphism. We can instead do something more transparent and easy but equivalent. Before doing that formally let's see the transition maps of these potential charts. If $(U, \phi)$ and $(V, \psi)$ are both charts on M,

$$\begin{tikzcd} & T(U\cap V) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\ \varphi(U\cap V)\times\mathbb{R}^n \arrow[rr, "\tilde{\psi}\circ\tilde{\phi}^{-1}"] & & \psi(U\cap V)\times\mathbb{R}^n \end{tikzcd}$$

For a point $(p, v) \in U \cap V$, the tangent vector $v$ at $p$ in the bases induced by the charts read as,

$$a_i \frac{\partial}{\partial x_i}\bigg|_p = b_i \frac{\partial}{\partial y_j}\bigg|_p \in T_pM. \tag{3.4}$$

The $a_i$ and $b_i$ are realted by the change of coordinates through the Jacobian matrix of the transition function in M. So, the transition maps on TM are simply given by,

$$\left(\tilde{\psi} \circ \tilde{\phi}^{-1}\right)(x, a) = \left(\psi \circ \phi^{-1}(x), J_{\psi\circ\phi^{-1}}(x)a\right). \tag{3.5}$$

and is thus smooth (from the smoothness of $\psi \circ \phi^{-1}$).

3.1.1. *Topology on the Tangent Bundle.* We will now formally give the topology on $TM$. Let $\{(U_\alpha, \phi_\alpha) | \alpha \in \mathbb{N}\}$ be a countable atlas of M, and $\{B_{\alpha\beta} | \alpha, \beta \in \mathbb{N}\}$ be a countable basis of $\phi_\alpha(U_\alpha) \times \mathbb{R}^n$. One can always get hold of such countable atlas because $M$ has a countable basis.

**Claim 3.1.** $\{\tilde{\phi}_\alpha^{-1}(B_{\alpha,\beta}) | \alpha, \beta \in \mathbb{N}\}$ *is a basis of a topology on* $TM$*.*

*Proof.* We need to show that the collection covers $TM$ and also that there is another set in the collection in an intetion of sets in the collection.

(1) $\bigcup_\beta \tilde{\phi}_\alpha^{-1}(B_{\alpha\beta}) = \bigcup_\alpha TU_\alpha = TM$, because $B_{\alpha\beta}$ cover $\phi_\alpha(U_\alpha) \times \mathbb{R}^n$.

(2) Let $V = \tilde{\phi}_\alpha^{-1}(B_{\alpha\beta}) \cap \tilde{\phi}_\gamma^{-1}(B_{\gamma\delta}) \subseteq TU_\alpha \cap TU_\gamma \subseteq TM$ be the intersubsection of two potential basis sets, then it suffices to show $\tilde{\phi}_\alpha(V)$ is open in $\tilde{\phi}_\alpha(U_\alpha) \times \mathbb{R}^n$.

$$\begin{aligned}\tilde{\phi}_\alpha(V) &= B_{\alpha\beta} \cap \tilde{\phi}_\alpha \left(TU_\alpha \cap \phi_\gamma^{-1}\left(B_{\gamma\delta}\right)\right) \\ &= B_{\alpha\beta} \cap \tilde{\phi}_\alpha \circ \tilde{\phi}_\gamma^{-1} \left(\tilde{\phi}_\gamma \left(TU_\alpha \cap TU_\gamma\right) \cap B_{\gamma\delta}\right) \\ &= \ldots \cap .. \left(\left(\phi_\gamma \left(U_\alpha \cap U_\gamma\right) \times \mathbb{R}^n\right) \cap B_{\gamma\delta}\right) \end{aligned} \tag{3.6}$$

since $U_\alpha \cap U_\gamma$ is open in $M$, we get $\tilde{\phi}_\alpha(V)$ to be open.

□

**Claim 3.2.** *The projection map $\pi : TM \to M$ given by $\phi(p, v) = p$ is a continous map.*

*Proof.* Let $U \subseteq M$ be open, we need to show that $\pi^{-1}(U)$ is open in $TM$. It is enough to show that for each $U_\alpha$ chart of $M$, $\pi^{-1}(U_\alpha \cap U)$ is open, since such sets cover $\pi^{-1}(U)$. Let $(p, w) \in \pi^{-1}(U_\alpha \cap U)$. Then $\exists B_{\alpha\beta}$ s.t,

$$\tilde{\phi}_\alpha((p, w)) \in B_{\alpha\beta} \subseteq \tilde{\phi}_\alpha(U_\alpha \cap U) \times \mathbb{R}^n.$$

$$\implies (p, w) \in \tilde{\phi}_\alpha^{-1}(B_{\alpha\beta}) \subseteq \pi^{-1}(U_\alpha \cap U) \implies \pi^{-1}(U_\alpha \cap U) \text{ is open.}$$

□

**Claim 3.3.** *The topology on $TM$ generated by this basis is second countable and Hausdorff.*

*Proof.* The countability of the basis of $TM$ is inherited by defition. We will show that $TM$ is indeed Hausdorff.

Let $(p, v) \neq (q, w) \in TM$.

(1) If $p \neq q$, then $\exists U \ni p, V \ni q \subseteq M$ s.t. $U \cap V = \emptyset$. $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open in $TM$, since $\pi$ is continous. Thus $\pi^{-1}(U) \cap \pi^{-1}(V) = \emptyset$.

(2) If $p = q, v \neq q$, Let $U_\alpha \ni p$. Since $\tilde{\phi}$ is a bijection, $\tilde{\phi_\alpha}((p, v)) \neq \tilde{\phi_\alpha}((p, w))$. So in $\tilde{\phi}_\alpha(U_\alpha) \times \mathbb{R}^n$ there exist $B_{\alpha\beta}, B_{\alpha\gamma}$ containing $\tilde{\phi_\alpha}(p, v), \tilde{\phi_\alpha}(p, w)$ respectively such that $B_{\alpha\beta} \cap B_{\alpha\gamma} = \emptyset$. Thus $\tilde{\phi_\alpha}^{-1}(B_{\alpha\beta}), \tilde{\phi_\alpha}^{-1}(B_{\alpha\gamma})$ are the required sets.

□

We finally have,

**Theorem 3.4.** *$TM$ is a 2n-dimensional smooth manifold with atlas given by,*

$$\left\{\left(TU_\alpha, \tilde{\phi_\alpha}\right) \mid \alpha \in N\right\} \tag{3.7}$$

*Proof.* The only thing left is to show that $\tilde{\phi}$ is a homeomorphism. The compatibility of these potential charts follows from the smoothness of the transition functions 3.5. From the defintion 3.3 it is clear that $\tilde{\phi}$ is a bijection. Continuity follows from the fact that inverse image of all the basic sets of $\tilde{\phi_\alpha}(U_\alpha) \times \mathbb{R}^n$ are defined to be the basic open sets of $TM$. And since all the basic sets in $TM$ are precisely the union of $\phi^{-1}(\tilde{B_{\alpha\beta}})$, we have $\tilde{\phi}$ to be an open map.

□

$TM$ is called the **Tangent Bundle** of the manifold M. The projection map $\pi : TM \to M$ is a surjective smooth map, since locally the transition functions are just the projection maps from $\mathbb{R}^{2n}$ onto $\mathbb{R}^n$ which are smooth. Infact $\pi$ is a smooth submersion. Tangent bundles are the special case of vector bundles where every preimage of the projection map are vector spaces attached to a point.

**Definition 3.5.** A smooth vector bundle of rank $k$ over a manifold $M$ is a triple $(V, M, \pi)$ where

(1) V is a smooth manifold and $\pi : V \to M$ is a smooth surjection.
(2) For each $p \in M$, $V_p = \pi^{-1}(p)$ is a vector space of dim. k. It's called the fibre at $p$
(3) (Local Trivialization) For any $p \in M$, there is a neighborhood $U \subseteq M$, and a diffeomorphism, $h : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "h"] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[dl, "proj._U"] \\ U & \end{tikzcd}$$

and also $h : V_q \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism for each $q \in U$.

*Remark* 3.6. dim V = dim M + k. And $\pi$ is a submersion, because $proj._U \equiv \tilde{\pi}$ is a submersion. Indeed for any tangent vector $v$ at $p$ in $T_pU$, we can define $w \in T_{\tilde{\pi}^{-1}(p)}\left(U \times \mathbb{R}^k\right)$ as $w(f) = v(f\,|_{U\times\{0\}})$, for all $f \in C^\infty(U)$. The diffeomorphism $h$ ensures submersion of $\pi$ as well.

*Remark* 3.7. $h : \pi^{-1} \to U \times \mathbb{R}^k$ is called a local trivialization,

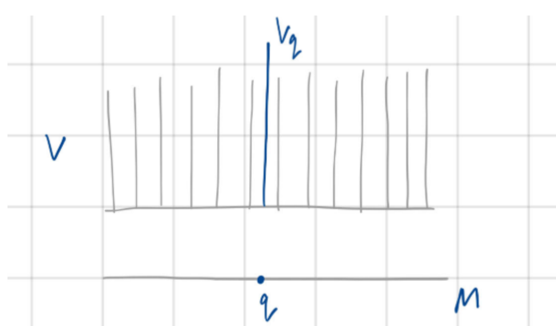



FIGURE 21. Local Trivialization of V

Some examples which will clarify in steps the different parts of definitions.

**Examples.**

1. **Trivial Vector Bundle**

$M$ smooth manifold. Then $\left(M \times \mathbb{R}^k, M, \pi_M\right)$ is a vector space bundle of rank $k$ over M. The projection map,

$$\pi_M : M \times \mathbb{R}^k \to M$$
$$\pi_M(p, x) = p$$

is indeed a smooth surjection. And for each point $p \in M$, we have $V_p = \mathbb{R}^k$ a $k - dim$ vector space. And the identity map of $M \times \mathbb{R}^k$ is the local (global!) trivialization. This is the **trivial vector bundle** of rank $k$ over M.

2. **Tangent Bundle**

If $M$ is an n-manifold, then $TM$ is a vector bundle of rank $n$ over $M$.

$$\pi : TM \to M, \quad \pi(p, v) = p,$$

$\pi$ is a smooth surjection as we already noted in the last subsection. And $\pi^{-1}(p)$ is precisely the tangent space at $p, T_pM$ an $n-$dimensional vector space. And recall that our charts on $TM$ are precisely in the form of local trivialization. And charts indeed define diffeomorphisms.

$$\begin{tikzcd}
TU \arrow[r, "h"] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[dl, "\tilde{\pi}"] \\
U &
\end{tikzcd}$$

And indeed, $T_pU \overset{h}{\cong} p \times \mathbb{R}^n$.

Now, for a more non-trivial example.

3. **Möbius Bundle**

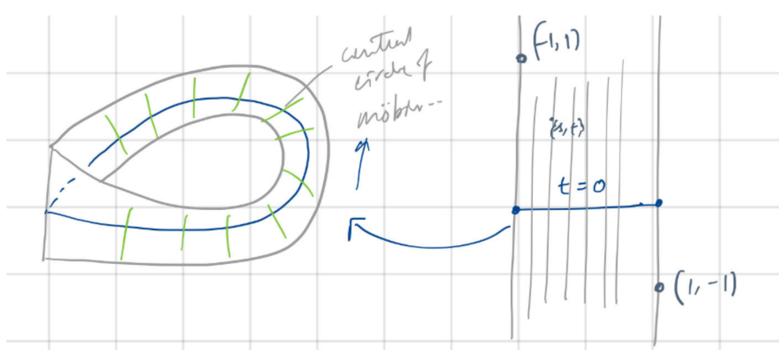



FIGURE 22. The mobius strip can be realised as the bundle over $S^1$. Note the $\mathbb{R}$ attached to every point on the central circle.

The intuition is clear, let's make it formal. Mobius strip is defined as,

$$M = [-1,1] \times \mathbb{R}/\sim, \quad (-1,t) \sim (1,-t)\,.$$

$$\begin{tikzcd}
\tilde{\pi} : [-1,1] \times \mathbb{R} \arrow[rr, "\tilde{\pi}"] \arrow[dr, "q"'] & & S^1 \\
& M \arrow[ur, "\pi"'] &
\end{tikzcd}$$

$\tilde{\pi}(s,t) = e^{\pi i s}$ and $\tilde{\pi}(-1,t) = \tilde{\pi}(1,t')$ induces a continous map $\pi : M \to S^1$. Infact we also saw that, $M \cong S^1 \times \mathbb{R}/\sim$, with $(z,t) \sim (-z,-t)$. We will work with this quotient space to see the bundle structure. Note the following diagram commutes and induces the same map $\pi : M \to S^1$.

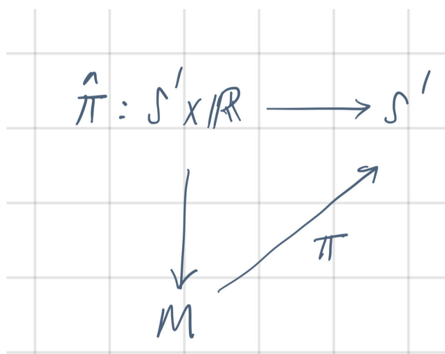



FIGURE 23. $\tilde{\pi}(z,t) = z^2$.

Let's verify the vector bundle strucutre in steps,

- $\tilde{\pi}$ is smooth $\implies$ $\pi$ is smooth.
- $\pi^{-1}(w) = \{[z,t] \mid z^2 = w\} = \{\pm z\} \times \mathbb{R}/\sim \cong \mathbb{R}$

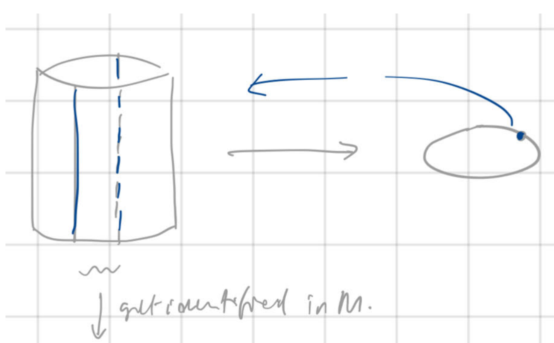



FIGURE 24

- Consider $U_+ = S^1 - \{-1\}, U_- = S^1 - \{1\}$. Then $\tilde{\pi}^{-1}(U_+) = (S^1 - \{\pm i\}) \times \mathbb{R}$. Define,
$$V_+ = \left\{z \in S^1 \mid Re z > 0\right\} \times \mathbb{R} \overset{o}{\subset} \tilde{\pi}^{-1}(U_+)$$
Note that $q : S^1 \times \mathbb{R} \to M$ is a local diffeomorphism so we immediately have the diffeomorphism $q : V_+ \to q(V_+) = \pi^{-1}(U_+)$. But $V_+$ is diffeomorphic to $U_+ \times \mathbb{R}$ given by,
$$f : V_+ \to U_+ \times \mathbb{R}, \quad f(z,t) = (z^2, t).$$
We thus have the following local trivialization on $M$.
$$h = f \circ q^{-1} : \pi^{-1}(U_+) \to U_+ \times \mathbb{R}$$

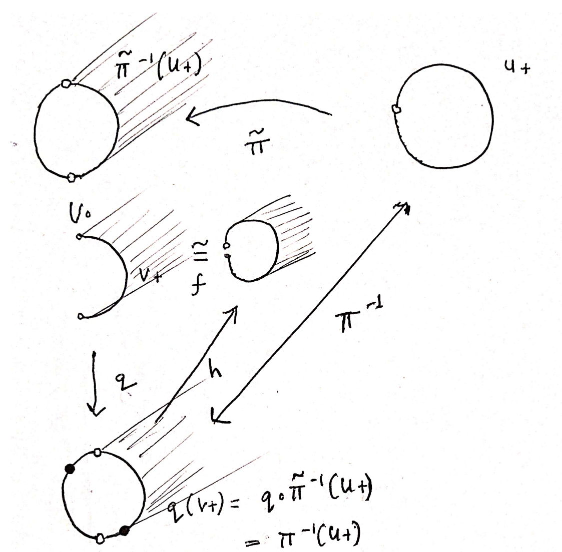



FIGURE 25. Möbius strip as a non-trivial bundle over $S^1$

Similarly, we have trivializations with $U_-$. We already saw that the fibers for each point in $U_+$ are of form $\{\pm z\} \times \mathbb{R}/\sim$ ($\cong \mathbb{R}$) and are indeed mapped to $\{z\} \times \mathbb{R}$ in $U_+ \times \mathbb{R}$ under $h$ linearly, and isomorphically.

So, Möbius strip is a bundle over $S^1$!

1. Fix the diffeomorphism between M and $S^1 \times \mathbb{R}/\sim$.

2. Show that both induce the same $\pi$ map. These questions are not much relevant to present discussion, these must have been clear in the previous subsections. Make sure you fix them in time.

4. **Tautological (line) Bundle of $\mathbb{R}P^n$**

$$\mathbb{R}P^n = \frac{\mathbb{R}^{n+1} - \{0\}}{\sim}, \quad x \sim cx, c \in \mathbb{R}.$$

We define,

$$L_n = \{([x], v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in span(x)\}. \tag{3.8}$$

This is a bundle over $\mathbb{R}P^n$ with,

$$\pi : L_n \to \mathbb{R}P^n, \quad \pi([x], v) = [x] \tag{3.9}$$

The fibers are given by, $\pi^{-1}([x]) = [x] \times span(x) \cong \mathbb{R}$. Define the topology on $L_n$ by demading $\pi$ is continous. Define the charts as follows,

$$\begin{aligned} \phi : \pi^{-1}(U_i) &\to \phi_0(U_i) \times \mathbb{R} \subseteq R^{n+1} \\ ([l], v) &\mapsto \left(\frac{l_0}{l_i}, \ldots, \frac{\hat{l}_i}{l_i}, \ldots, \frac{l_n}{l_i}, v_i\right) \end{aligned} \tag{3.10}$$

Injection: $\phi([l], v) = \phi([m], w) \iff [l] = [m] \& v_i = w_i \implies v = w$.
Surjection: If $(m_1, \ldots, m_{n+1}) \in \mathbb{R}^{n+1}$, then,

$$\phi([m_1, \ldots, m_{i-1}, 1, m_i, \ldots, m_n], m_{n+1}(m_1, \ldots, 1, m_i, \ldots, m_n)) = (m_1, \ldots, m_{n+1}).$$

The smoothness of transition functions is easy to check. The local trivialization is simply,

$$\begin{aligned} \phi : \pi^{-1}(U_i) &\to U_i \times \mathbb{R} \\ ([l], v) &\mapsto ([l], v_i) \end{aligned}$$

the composition of charts (diffeomorphisms) on $L_n$ and the charts on $\mathbb{R}P^n$.

$L_n$ is a vector bundle of rank 1 over $\mathbb{R}P^n$. Rank 1 vector bundles are also called line bundles.

Infact, $\mathbb{R}P^n$ is a regular submanifold of $L_n$ which can be seen from the charts on $L_n$. $V_i = \{([l], 0) \mid l_i \neq 0, l \in \mathbb{R}^{n+1}\} \subseteq \pi^{-1}(U_i) = \phi_i^{-1}(\mathbb{R}^n \times \{0\}) \cong U_i$.

$L_1$ is exactly the Möbius bundle over $S^1$. because note that $\mathbb{R}P^1 \cong S^1$.

### 3.2. **Homomorphisms of Vector Bundles.**

**Definition 3.8.** Let $\pi : V \to M$ and $\pi' : V' \to N$ be two vector bundles. A homomorphism $\tilde{f} : V \to V'$ is a smooth map s.t. $\exists f : M \to N$ for which the following diagram commutes,

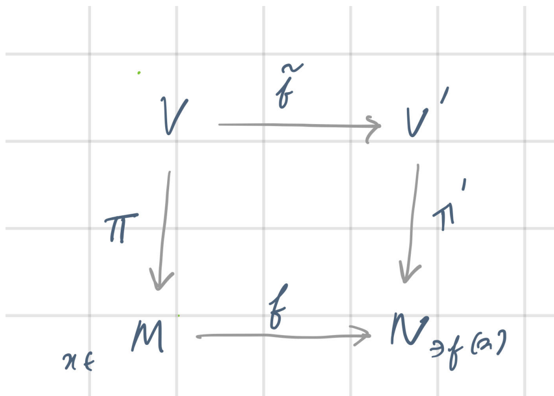



FIGURE 26. Vector Bundle Homomorphism

i.e $\tilde{f}(V_x) \subseteq V'_{f(x)}$. And $\tilde{f} : V_x \to V'_{f(x)}$ is a fiber to fiber linear map.

**Definition 3.9.** Let $\pi : V \to M$ and $\pi' : V' \to M$ be two vector bundles over $M$. And isomorphism $\tilde{f} : V \to V'$ is a diffeomorphism s.t the following diagram commutes,

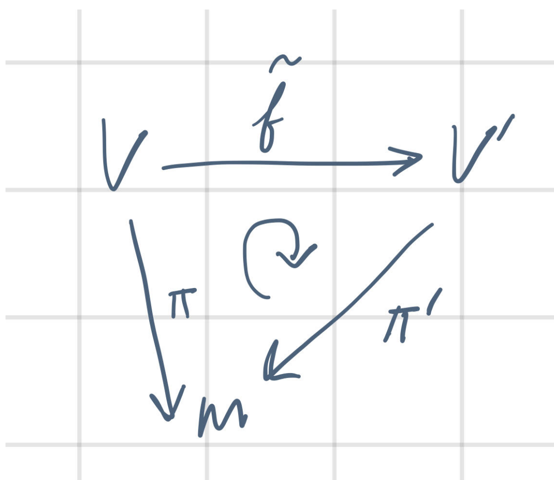



FIGURE 27. Vector Bundle Isomorphism

i.e $\tilde{f}(V_x) = V'_x$. And $\tilde{f} : V_x \to V'_x$ is a fiber to fiber linear isomorphism.

**Examples.**
1. $F : M \to N$ is a smooth map. Define,

$$\begin{aligned} F_* &: TM \to TN \\ &(p, v) \mapsto (F(p), F_{*p}(v)) \end{aligned} \tag{3.11}$$

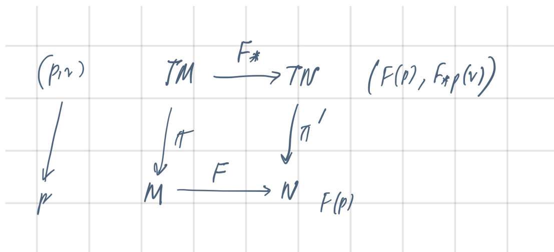



FIGURE 28. Tangent Bundle Homomorphism

$F_*$ is a linear map because $F_{*p}$ is linear for each $p \in M$. We need to show $F_*$ is smooth. Let $U, \phi$ be a chart of $M$ and $V, \psi$ be a chart of N, st. $F(U) \subseteq V$. Further, let $\left(TU, \tilde{\phi}\right)$ be a chart of $TM$,

$$\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^m$$

and $\left(TV, \tilde{\psi}\right)$ be a chart of $TN$,

$$\tilde{\psi} : TV \to \psi(V) \times \mathbb{R}^n$$

Then, $\tilde{\psi} \circ F_* \circ \tilde{\phi}^{-1}(x, c) = (\psi \circ F \circ \phi^{-1}(x), J_{\psi \circ F \circ \phi^{-1}}(x)\, c)$ is indeed smooth. So, $F_*$ is smooth and is a vector bundle homomorphism between $TM$ and $TN$.

2. In the above example, if $F$ is an embedding, then so is $F_*$.

3. If $M \subseteq N$ is regular submanifold, then $TM \subseteq TN$ is also a regular submanifold. If $U, \phi$ is an adapted chart of M, then $TU, \tilde{\phi}$ is an adapted chart of $TM$. As special cases of we have the following two examples. 4. $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, with $\left(p, c_i \frac{\partial}{\partial x_i}|_p\right) \mapsto (p, (c_1, \ldots, c_n))$ is an isomorphism of vector bundles over $\mathbb{R}^n$. So, if $M \subseteq \mathbb{R}^n$ is a regular submanifold, then $TM \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a regular submanifold.

5. $S^n \subseteq \mathbb{R}^{n+1}$. Then

$$TS^n = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, v \rangle = 0\}$$

with $\pi : TS^n \to S^n, \quad \pi(x, v) = x$.

If $n = 2$,

$$TS^2 = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1, v_1 x_1 + v_2 x_2 + v_3 x_3 = 0\}.$$

3.3. **Sections.** There are many kind of subsections: $L_1$, $L_2$, Smooth, Measurable...

**Definition 3.10.** Let $\pi : V \to M$ be a smooth vector bundle. A subsection $s : M \to V$ is a map s.t. $\pi \circ s = id_M$, i.e. $s(x) \in V_x$.

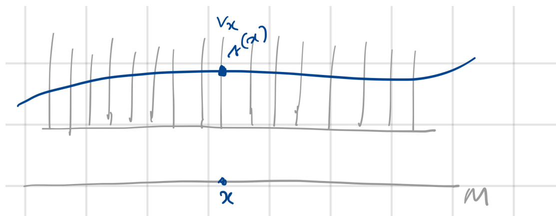



FIGURE 29. Section of a vector bundle

We say that $s$ is a smooth subsection, if $s$ is a smooth map.

**Examples.**

1. if $V = M \times \mathbb{R}^k \xrightarrow{\pi} M$, the a smooth subsection $s : M \to M \times \mathbb{R}^k$ is basically $s(x) = (x, f(x))$, where $f : M \to \mathbb{R}^k$ is a smooth map. So, $s$ on a trivial vector bundle is determined entirely by the $f$.

2. If $\pi : V \to M$ is a vector bundle and $0_x$ is zero of $V_x$. The subsection $z : M \to V, z(x) = 0_x$ is a smooth subsection.

Let $(U, \phi)$ be a chart of $M$ such that, $h : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization. Then,

$$U \xrightarrow{z} \pi^{-1}(U) \xrightarrow{h} U \times \mathbb{R}^k, \quad h(z(x)) = (x, 0)$$

Since $h \circ z : U \to U \times \mathbb{R}^k$ is smooth, $z$ is also smooth.

**Definition 3.11.** A local subsection of $\pi : V \to M$ is a subsection over some open subset $U$ of $M$,

$$s : U \to \pi^{-1}(U) = V|_U$$

which is a smooth map s.t. $\pi \circ s = Id_U$.

We denote the set of all local subsections of vector bundle $V$ by,

$$\Gamma(V, U) = \{s : U \to V|_U \mid \pi \circ s = Id_U, s \text{ is smooth}\}$$

And the set of all global subsections by $\Gamma(V) := \Gamma(V, M)$.

**Examples.** The set of all global subsections of trivial vector bundle is just the set of all smooth functions from $M$ to $\mathbb{R}^k$. The Zero subsection, z is a global subsection for any vector bundle. Infact, $z$ is an embedding of M in V.

**Exercise.** Show that any subsection $s : M \to V$ is an embedding.

**Proposition 3.12.** *Let $\pi : V \to M$ be a vector bundle.*

(1) *If $s_1, s_2 \in \Gamma(V)$, then $s_1 + s_2 \in \Gamma(V)$.*
(2) *If $s \in \Gamma(V), f \in C^\infty(M)$ then $fs \in \Gamma(V)$.*
(3) *$\Gamma(V)$ is a module over $C^\infty(M)$.*

Infact this is a projective module and is an algebraic geometric way of defining vector bundles.

*Proof.* We define $(s_1 + s_2)(x) = s_1(x) + s_2(x) \in V_x$. We want to show that $s = s_1 + s_2$ is a smooth map. For $p \in M$, choose an open set $U \subseteq M$ containing $p$ s.t. we have the local trivialization,

$$h : V|_U \to U \times \mathbb{R}^k.$$

$s|_U$ is smooth $\iff$ $h \circ s|_U$ is smooth. $h \circ s_i(x) = (x, f_i(x))$ as a subsection of a trivial vector bundle where $f_i : U \to \mathbb{R}^k$ is smooth. Then $h \circ s(x) = (x, f_1(x) + f_2(x))$ is smooth. The multiplication with $f \in C^\infty(M)$ is similarly pointwise and we thus have a module structure over $C^\infty(M)$. $\square$

We saw two line bundles over $S^1$,

(1) $\pi : M \to S^1$ the Möbius Bundle.
(2) $\pi : S^1 \times \mathbb{R} \to S^1$ the trivial line bundle.

**Proposition 3.13.** *$M$ and $S^1 \times \mathbb{R}$ are not isomorphic as vector bundles.*

*Proof.* Suppose $h : V = S^\times \mathbb{R} \to M$ is an isomorphism. Then $h$ should take $z_V(S^1)$ to $z_M(S^1)$.

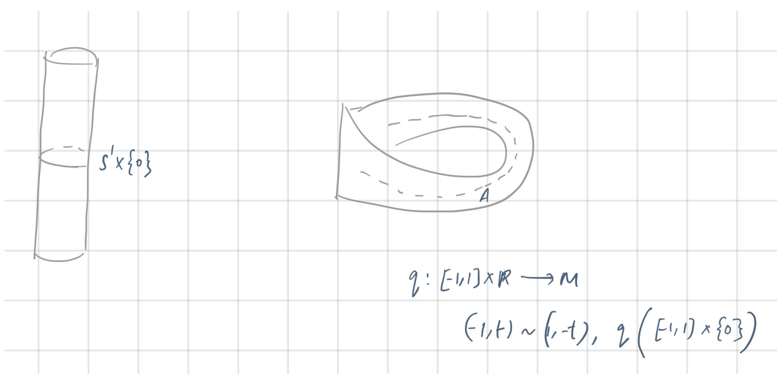



FIGURE 30. The image of zero subsections of V and M.

Let $h(S^1 \times \{0\}) = A$. $S^1 \times R - S^1 \times \{0\} = S^1 \times (\mathbb{R} - \{0\})$ is disconnected where as, $M - A$ is connected, which contradicts the fact that $h$ is a contradiction.

**Claim 3.14.** *$M - A$ is connected.*

*Proof.* It is infact path connected.

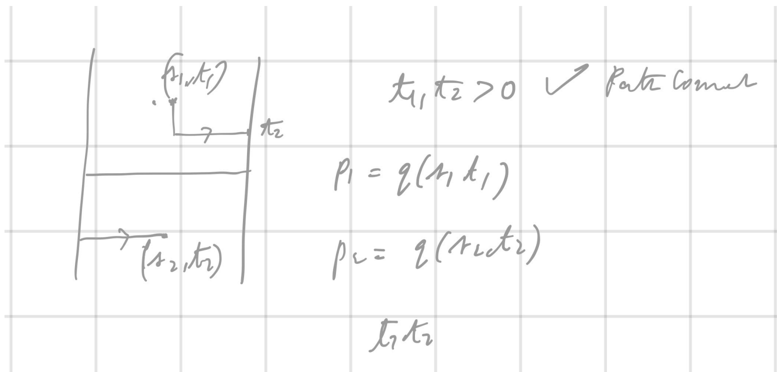



FIGURE 31. M-A is path connected.

∎

So, Möbius bundle and the trivial bundle over $S^1$ are not isomorphic as vector bundles. ∎

We can talk about smoothly varying basis on (fibers of) vector bundle.

## 3.4. Frames.

**Definition 3.15.** $\pi : V \to M$ be a vector bundle of rank $k$, and $U \subseteq M$ an open subset. If $s_1, \ldots, s_k \in \Gamma(V, U)$ such that $s_1(x), \ldots, s_k(x)$ forms a basis of $V_x$ $\forall x \in U$, then $\{s_1, \ldots, s_k\}$ is called a local frame of $V$ on U.

*Proof.* $\Longrightarrow$ : If $M \times \mathbb{R}^k \xrightarrow{h} V$ is an isomorphism, then define $s_i : M \to V, s_i(x) = h(x, e_i)$. $s_1(x), \ldots, s_k(x)$ form a basis. ∎

$\Longleftarrow$ : Let $s_1, \ldots, s_k \in \Gamma(V)$ be a global frame. Then define,

$$h : M \times \mathbb{R}^k \to V$$
$$h(x, c) = c_1 s_1(x) + \cdots , c_k s_k(x) \in V_x$$

$h : \{x\} \times \mathbb{R}^k \to V_x$ is a linear isomorphism.

**Claim 3.16.** *$h$ is a diffeomorphism.*

*Proof.* $h$ is clearly a bijection. And it suffices to show that $h$ is a smooth immersion (by inverse function theorem). Let $(U, \phi)$ be a chart of $M$, s.t. $V|_U \xrightarrow{g} U \times \mathbb{R}^k$ is a trivialization.

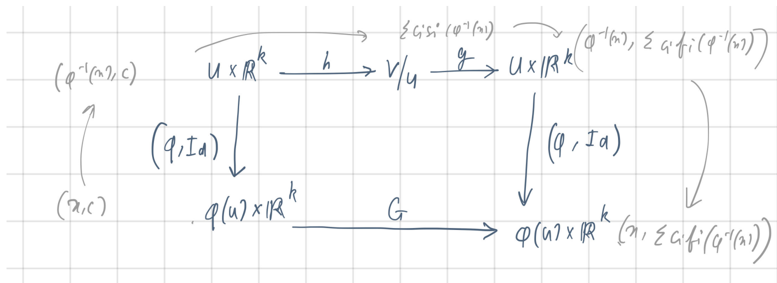


FIGURE 32

$g(s_i(x)) = (x, f_i(x))\,,\ f_i : U \to \mathbb{R}^k$ is smooth. Then, $G(x,c) = (x, \sum c_i f_i(\phi^{-1}(x))$ is smooth with the jacobian,

$$J_G(x,c) = \begin{pmatrix} I & 0 \\ * & f_1\phi^{-1}(x)\cdots f_k\phi^{-1}(x) \end{pmatrix} \tag{3.12}$$

This is invertible because $\{f_i\}$ are linearly independent. Thus $h$ is a smooth immersion, and a diffeomorphism. □

**Remark.** Any vector bundle is locally trivial. So locally any subsection can be written as a function. It is indeed *remark*able that the local trivialization can now be written in terms of the local frames.

Infact this will give us a criterion for checking smoothness of subsections via local frames.

**Proposition 3.17.** *Suppose $s_1, \ldots, s_n \in \Gamma(V, U)$ which forms a frame of $V$ on $U$. Then a subsection $t : U \to V|_U$ is smooth $\iff$ $\exists a_1, \ldots, a_n \in C^\infty(U)$ s.t. $t = a_1 s_1 + \ldots + a_n s_n$.*

*Proof.* $\implies$: Consider the local trivialization $h : U \times \mathbb{R}^n \to V|_U$.
Then $h(p,c) = \sum_{i=1}^n c_i s_i(p)$ because $\{s_i\}$ is a frame on U. Note that a subsection from $U$ to $U \times \mathbb{R}^k$ is given by $s(p) = (p, a_1(p), \ldots, a_n(p))$. So, any subsection from $U$ to $V|_U$ is given by $h \circ s(p) = t(p) \implies t = a_1 s_1 + \ldots + a_n s_n$.

The other direction is trivally true since all $s_i$ are smooth. □

So we can use the trivialization cover of the vector bundle and the above result to show smoothness of subsection on $M$.

## 3.5. **Vector Fields.**

**Definition 3.18.** Let $M$ be a smooth manifold. A vector field on $M$ is a subsection of $TM$. A vector field $X : M \to TM$ is called smooth if it is a smooth subsection.

We denote the set of all vector fields on $M$ as, $\mathfrak{X}(M) := \Gamma(TM)$. $\mathfrak{X}(M)$ is a $C^\infty(M)$ module.

**Examples.**

1. What is a vector field on $\mathbb{R}^n$?

$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$$

If $X : \mathbb{R}^n \to T\mathbb{R}^n$ is a vector field then $X(p) = (p, v(p))$ where $v : \mathbb{R}^n \to \mathbb{R}^n$ is $C^\infty$. So a vector field $X$ is determined by $v \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

2. If $U \subseteq \mathbb{R}^n$, then $\mathfrak{X}(U) \cong C^\infty(U, \mathbb{R}^n)$.

3. $v : \mathbb{R}^2 \to \mathbb{R}^2$, $v(x, y) = (-y, x)$. Basically $X(x, y) = \left(x, y, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$. We often ditch the $x, y$ part of the vector field and simply write, $X(x, y) = (-y, x)$ or as an element of the tangent space which is subspace of $\mathbb{R}^n$.

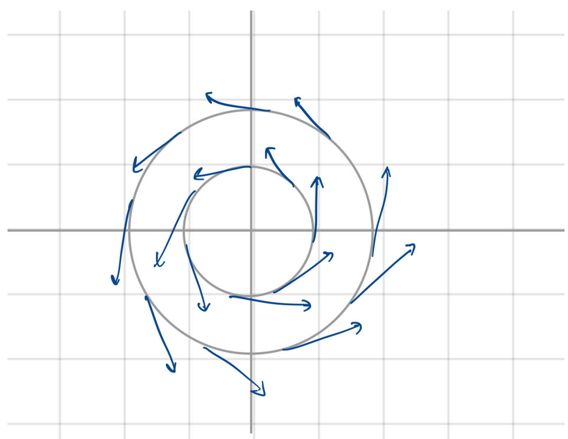


FIGURE 33. Curl Vector field in $\mathbb{R}^2$

4. [Gradient Vector Field]
If $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^\infty(\mathbb{R}^n)$ function,

$$\nabla f = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) : \mathbb{R}^n \to \mathbb{R}^n \tag{3.13}$$

is called the gradient vector field of $f$.

If $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = xy$ then the gradient vector field of $f$ evaluated at $f^{-1}(1)$ is,

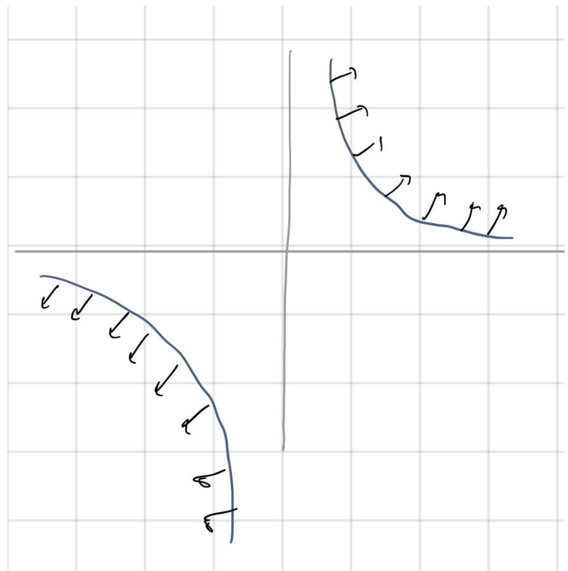


FIGURE 34. Gradient Vector field for $f(x, y) = xy$, evaluated at $f^{-1}(1)$.

If $M \subseteq \mathbb{R}^n$ is a regular submanifold, then $TM \subseteq T\mathbb{R}^n \cong \mathbb{R}^n \cong \mathbb{R}^n$ is a regular submanifold. If $X : M \to TM$ is a vector field, then $X$ gives us a smooth function, $X(p) = (p, v(p))$ where $v \in C^\infty(M, \mathbb{R}^n)$ and $v(p) \in T_p(M) \subseteq \mathbb{R}^n$.

So, vector fields on $M$ are given by smooth functions $v : M \to \mathbb{R}^n$ s.t. $v(p) \in T_pM$ $\forall p \in <$. Let's look at some examples of such cases.

5. Consider $S^1$ a reg. sub manifold of $\mathbb{R}^2$, and $X \in \mathfrak{X}(S^1)$, $X(x, y) = (-y, x)$. We see that $X(p) \neq 0$ $\forall p \in S^1$. Since $TS^1$ is a rank 1 vector bundle, $X$ being $\neq 0$ is a frame of $TS^1$. Thus $TS^1$ is the trivial vector bundle $TS^1 \cong S^1 \times \mathbb{R}$!

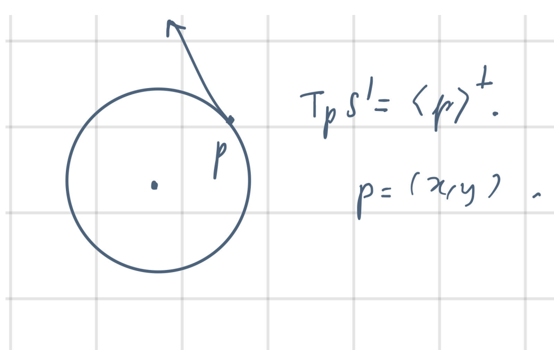



FIGURE 35. Non vanishing Vector Field on $S^1$ which is a global frame of $TS^1$.

**Remarks.** Indeed at any point on $S^1$ the fibers are $R^1$ but it is non-trivial that $TS^1$ is globally trivial! $TS^1$ being a line bundle, and the above $X$ a non-vanishing vector field gave us a global frame such that the entire tangent bundle is trivialized, or formally called "parallelizable". It is not true that any $TS^n$ is trivial. Infact $S^n$ is parallelizable only for $n = 1, 3, 7$. We note that non-vanishing vector fields are important to analyse the parallelizability of manifolds (since they can potentially give us a global frame). Infact, the tangent bundle of any Lie Group is trivial. $S^3$ is a lie group and $TS^3$ is trivial.

6. $S^2 \subseteq \mathbb{R}^3$. And consider $X : S^2 \to \mathbb{R}^3,\ X(x, y, z) = (-y, x, 0) \in \langle x, y, z\rangle^\perp = T_{(x,y,z)}S^2$. This vector field vanishes at the north and south poles. Infact this is related to the topology of $S^2$. The Eucler characteristic of $S^2$ is 2, and see Pooincaré Hopf Theorem which relates the (zero) index set of vector fields to Euler Characteristic! A special case of this is the Hairy Ball Theorem. $S^{2n}$ has no continous non-vanishing vector field!

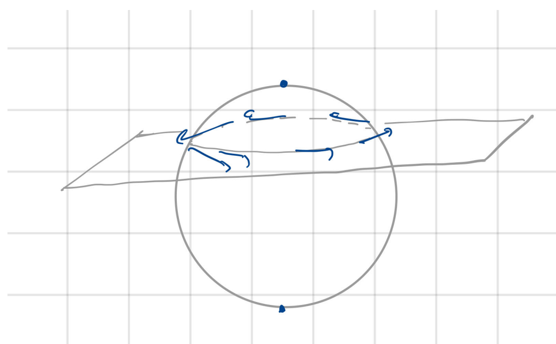


FIGURE 36. Vector field $X(x, y, z) = (-y, x, 0)$ on $S^2$ which vanishes at the poles.

This vector field is essentially a rotation of $S^2$. Infact vector fields give nice diffeomorphisms of the manifold as we will soon see.

If $(U, \phi)$ is a chart of $M, \phi = (x_1, \ldots, x_n)$ then $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is a local frame for $TM$ on $U$.

$$TU \xrightarrow{\tilde{\phi}} \phi(U) \times \mathbb{R}^n \xrightarrow{(\phi^{-1}, Id)} U \times \mathbb{R}^n$$

$$\tilde{\phi}\left(p, c_i \frac{\partial}{\partial x_i}|_p\right) = (\phi(p), (c_1, \ldots, c_n)) \tag{3.14}$$

**Proposition 3.19.** *If $X \in \mathfrak{X}(M)$, then $X|_U = \sum a_i \frac{\partial}{\partial x_i}$, where $a_i \in C^\infty(U)$.*

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}|_p \tag{3.15}$$

Basically vector fields provide a smooth way to move between tangent spaces of points in the manifold.

**Examples.**

1. Consider $X \in \mathfrak{X}(S^1)$, where $X(x,y) = (-y,x)$ and the chart $(U_+, \phi_+)$. We want to write the local expression for $X|_{U_+}$ using the above proposition. Note that we have been using the fact that $S^1$ is a regular submanifold of $\mathbb{R}^2$, and used the subspace definition of the tangent space at each poi9nt of $S^1$ to write vector fields as functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. Now however the form 3.15 is true locally for all manifolds, and we now need to rewrite $X$ in this form.

We have,

$$\begin{aligned}
&\phi_+ : U_+ \to \mathbb{R} \\
&\phi(x,y) = \frac{x}{1-y}, \ \phi^{-1}(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right)
\end{aligned} \tag{3.16}$$

let $p = (x,y)$. Then,

$$X(x,y) = -y\frac{\partial}{\partial x}|_p + x\frac{\partial}{\partial y}|_p = a(p)\frac{\partial}{\partial \theta} \tag{3.17}$$

where $a(p)$ is $C^\infty(U_+)$ and $\frac{\partial}{\partial\theta}$ is the basis of $T_pU_+$. We want to find $a(p)$. The above can be written as,

$$\begin{aligned}
-y\frac{\partial f}{\partial x}|_p + x\frac{\partial f}{\partial y}|_p &= a(p)\frac{df}{d\theta}|_p \\
&= a(p)\phi^{-1}_{*\phi(p)}\left(\frac{d}{dt}\right)(f)\,|_p \\
&= a(p)\frac{d}{dt}\left(f\circ\phi^{-1}\right)(\phi(p)) \\
&= a(p)\frac{d\phi_1^{-1}}{dt}\left(\phi(p)\right)\frac{\partial f}{\partial x}|_p + a(p)\frac{d\phi_2^{-1}}{dt}\left(\phi(p)\right)\frac{\partial f}{\partial y}|_p
\end{aligned} \tag{3.18}$$

This gives, $a(x,y) = 1-y$, and we have,

$$X(x,y) = (1-y)\frac{\partial}{\partial\theta}|_{(x,y)} \tag{3.19}$$

2. We can repeat the same calculation for $S^2$ with $X(x,y,z) = (-y,x,0)$. In particular, with the chart

$\phi : U_+ \to \mathbb{R}^2$

$$\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \quad \phi^{-1}(s,t) = \left(\frac{2s}{1+s^2+t^2}, \frac{2t}{1+s^2+t^2}, \frac{s^2+t^2-1}{1+s^2+t^2}\right) \tag{3.20}$$

we have,

$$X(p) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z} = \sum \left(a(p)\frac{\partial \phi_1^{-1}}{\partial s} + b(p)\frac{\partial \phi_i^{-1}}{\partial t}\right)(\phi(p))\,\frac{\partial}{\partial x_i}|_p \tag{3.21}$$

3.5.1. *Vector Fields as Derivations.* If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$ then define the functions $Xf : M \to \mathbb{R}$ given by $Xf(p) = X(p)(f_p)$.

**Claim 3.20.** *$Xf$ is a smooth function.*

*Proof.* If $(U, x_1, \ldots, x_n)$ is a chart of $M$, then $X(p) = \sum a_i(p)\frac{\partial}{\partial x_i}|_p$ for some $a_i \in C^\infty(U)$. So,

$$\begin{aligned} X(p)(f_p) &= \sum a_i(p)\frac{\partial f}{\partial x_i}p \\ &= \sum a_i(p)\frac{\partial f \circ \phi^{-1}}{\partial x_i}(\phi(p)) \end{aligned} \tag{3.22}$$

Since, $f$ is smooth and $\phi$ is a diffeomorphism, $Xf|_U$ is smooth, i.e. $Xf \in C^\infty(M)$. □

**Example.**
Let $X \in \mathfrak{X}(\mathbb{R}^n)$, where $X = a_1\frac{\partial}{\partial x_1} + \cdots + a_n\frac{\partial}{\partial x_n}$, $a_i \in C^\infty(\mathbb{R}^n)$. Then,

$$Xf(p) = \sum a_i(p)\frac{\partial f}{\partial x_i}(p).$$

We can also consider the curl like vector field $X$ on $S^1$, and see that $Xf$ is indeed smooth, for $f \in C^\infty(S^1)$. This is clear from last subsection where we computed the local form of that vector field.

Hence, any vector field $X \in \mathfrak{X}(M)$ gives a map $X : C^\infty(M) \to C^\infty(M)$. And infact $X$ is a linear map, because $X(p)$ acts linearly on $C^\infty(M)$ as a tangent vector (derivation). The leibniz rule is also obeyed, $X(fg) = fXg + gXf$. hence, $X$ gives a derivation of $C^\infty(M)$. Infact we have the following results (which we may prov later. It involves usage of bump function).

**Theorem 3.21.** *Let $Der\,(C^\infty(M))$ be the set of linear maps, $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ which satisfy $\mathcal{D}(fg) = f\mathcal{D}g + g\mathcal{D}f$, $\forall f, g \in C^\infty(M)$. Then the map*

$$\mathfrak{X} \to Der\,(C^\infty(M))$$

*is a linear isomorphism.*

**Proposition 3.22.** *If $X$ is a vector field on $M$, $X$ is smooth $\iff \forall f \in C^\infty(M)$, $Xf$ is smooth.*

3.5.2. *Vector Field Pushforward.* Suppose $F : M \to N$ is a smooth map. Can we take $X \in \mathfrak{X}(M)$ and get some $Y \in \mathfrak{X}(N)$ from $X$ and $F$.

**Problems.**

1. F may not be surjective. 2. F may not be injective. In this case $p_1, p_2 \in M, F(p_1) = F(p_2) = q \in N$.

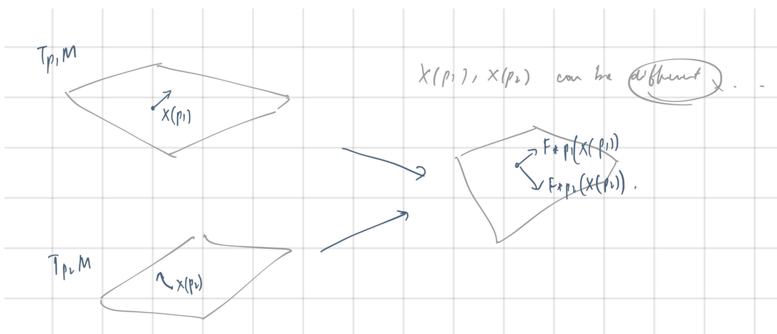



FIGURE 37. Pushforward of a vector field may not be well defined if $F$ is not injective.

If $F$ is a diffeomorphism, then we can define $F_*(X)$ as,

$$F_*(X)(q) = F_{*F^{-1}(q)} X\left(F^{-1}(q)\right) \ \forall q \in N. \tag{3.23}$$

Diffeomorphism is a strong condition to ask, but we can rectify the problem by demanding,

$$F_{*p}\left(X(p)\right) = F_{*q}\left(X(q)\right) \ \forall p, q \in M \ s.t. \ F(p) = F(q). \tag{3.24}$$

In this case we can define a vector field $Y$ on $N$ by,

$$Y(r) = F_{*p}(X(p)) \tag{3.25}$$

for any $p \in F^{-1}(r)$.

**Definition 3.23.** If $F : M \to N$ is surjective smooth map, then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called F-related if,

$$Y(F(p)) = F_{*p}(X(p)), \ \forall p \in M. \tag{3.26}$$

F-related vector fields can be hard to find, but we have an interesting result for quotient manifolds, obtained by action by a discreet group. Before stating that let's note a property of $F - related$ vector fields.

**Lemma 3.24.** *$X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ are $F-$related if and only if,*

$$(Yg) \circ F = X\,(g \circ F)\ \ \forall g \in C^\infty(N).$$

*Proof.*

$$M \xrightarrow{F} N \xrightarrow{Yg} \mathbb{R}$$

For $p \in M$,

$$\begin{aligned}(Yg)\circ F(p) &= Y(F(p))\left(g_{F(p)}\right)\\ &= F_{*p}\left(X(p)\right)\left(g_{F(p)}\right)\\ &= X(p)\left(\left(g\circ F\right)_p\right) = \left(X\left(g \circ F\right)\right)(p).\end{aligned}$$

So, $Y \curvearrowright C^\infty(N) \xrightarrow{F} C^\infty(M) \curvearrowleft X$. $\square$

**Proposition 3.25.** *If $G$ is a discrete group acting smoothly and properly discontinously on $M$, $N = M/G$ and $F : M \to N$ is the quotient map (which is also a local diffeomorphism!). Suppose $X \in \mathfrak{X}(M)$ is G-invariant vector field, i.e*

$$g_{*p}\left(X(p)\right) = X(gp)\ \forall g \in G, p \in M.$$

*Then there is a vector field $Y \in \mathfrak{X}(N)$ which is $F-$related to $X$.*

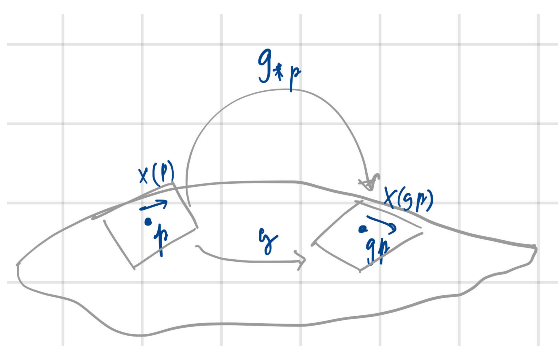



Figure 38. G-invariant vector field on M.

*Proof.* $r = F(p) = F(q) \iff q = gp$ for some $g \in G$. Then $X(q) = X(gp) = g_{*p}X(p)$ Then,

$$\begin{aligned}F_{*q}(X(q) &= F_{*q}\left(X(gp)\right)\\ &= F_{*q} \circ g_{*p}\left(X(p)\right)\\ &= F_{*p}\left(X(p)\right)\end{aligned}$$

This shows that $Y(F(p)) = F_{*p}(X(p)$ is well defined. And since $F$ is a local diffeomorphism, $Y$ is smooth. By abuse of notation, we write $Y = F_*X$. $\square$

**Examples.**
1. Consider the vector field $X \in \mathfrak{X}(\mathbb{R})$,

$$X(p) = \frac{d}{dt}|_p \text{ } or \text{ } X(p) = 1.$$

$\mathbb{Z}$ acts on $\mathbb{R}$ as $n \cdot t = t + n$. Then, $n_{*p}(X(p)) = 1$. So $X$ is a $\mathbb{Z} -$ *invariant* vector field on $\mathbb{R}$.

Other way to see this is to consider $\gamma(t) = p + t, \gamma'(0) = \frac{d}{dt}|_p$ and $n \circ \gamma(t) = p + t + n, n_* \gamma'(0) = \frac{d}{dt}|_{n+p}$.

Now, let $F : \mathbb{R} \to S^1$ be the quotient map, with $F(t) = (\cos t, \sin t)$. Then

$$\begin{aligned} F_{*p}\left(\frac{d}{dt}|_p\right) &= \frac{dF_1}{dt}\frac{\partial}{\partial x_1}|_{F(p)} + \frac{dF_2}{dt}\frac{\partial}{\partial x_2}|_F(p) \\ &= -\sin p \frac{\partial}{\partial x_1}|_{F(p)} + \cos p \frac{\partial}{\partial x_2}|_{F(p)} \\ &= \left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right)|_{F(p)} \end{aligned} \tag{3.27}$$

This is exactly the curl like vector field we have been discussing a lot on $S^1$!
2. Consider the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$,

$$(m, n) \cdot (s, t) = (s + m, t + n).$$

On $\mathbb{R}^2$ we have two vector fields which are $\mathbb{Z}^2$ invariant,

$$X_1 = \frac{\partial}{\partial s}, X_2 = \frac{\partial}{\partial t}.$$

Hence, if $q : \mathbb{R}^2 \to T^2 = S^1 \times S^1$ is the quotient map, then $Y_i = q_* X_i$ are the vector fields on $T^2$.

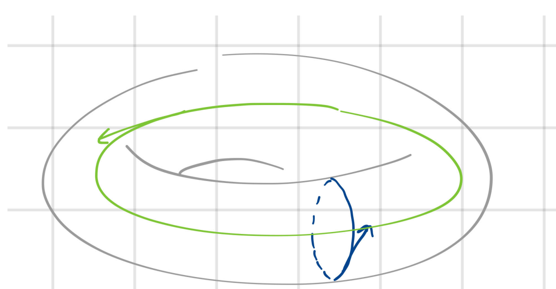


FIGURE 39. Vector Fields on $T^2$ which are $q-$*related* to $X_1, X_2$ on $\mathbb{R}^2$.

3. $C_2 \curvearrowright S^2$, $\rho(x) = -x$. Recall, $\frac{S^2}{C_2} \cong \mathbb{R}P^2$.

Consider $X \in \mathfrak{X}(S^2)$, $X(x) = (-x_2, x_1, 0) \equiv -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$ where $x = (x_1, x_2, x_3)$. Then $X(-x) = (x_2, -x_1, 0) = \rho_{*x} X(x)$.

Hence, $Y = q_* X$ will be a smooth vector field on $\mathbb{R}P^2$, which will have only one zero since the antipodal points are identified.

3.6. **Integral Curves.** Suppose $M$ is a smooth manifold and $X \in \mathfrak{X}(M)$. Then a curve $\gamma : (a, b) \to M$ is called an integral curve of $X$ if

$$\gamma'(t) = X(\gamma(t)).$$

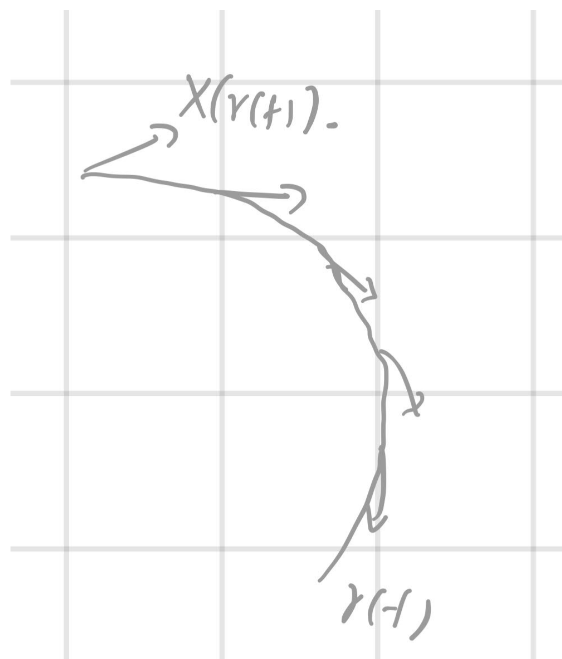



FIGURE 40. Integral Curve of $X$.

If $0 \in (a, b)$ and $\gamma(0) = p$, then we say that $\gamma$ is an integral curve starting at $p$ or $p$ is the initial point of $\gamma$.

**Examples.**

1. Consider $X \in \mathfrak{X}(\mathbb{R})$, $X(p) = \frac{d}{dt}|_p$. Then $\gamma(t) = p + t$ is an integral curve of $X(p)$.

2. Similarly if $X = \mathfrak{X}(\mathbb{R}^n)$ is a constant vector field $X(p) = a$, for some $a \in \mathbb{R}^n$, then $\gamma(t) = a + t$ is an integral curve passing through $p$.

3. Consider $X \in \mathfrak{X}(\mathbb{R}^2)$, $X(x, y) = (-y, x)$.

If $\gamma$ is the integral curve with $\gamma(0) = (x_0, y_0)$ then,

$$\gamma'(t) = (-\gamma_2(t), \gamma_1(t)) \implies \dot{\gamma}_1(t) = -\gamma_2(t); \dot{\gamma}_2(t) = \gamma_1(t) \implies \ddot{\gamma}_1(t) = -\gamma_1(t).$$

We need to solve the 2nd order ODE. This has a general solution:

$$\gamma_1(t) = a\cos t + b\sin t; \gamma_2(t) = a\sin t - b\cos t.$$

Imposing the initial condition $(x_0, y_0)$, we see that,

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

This is just rotation of $(x_0, y_0)$ by an angle $-t$.

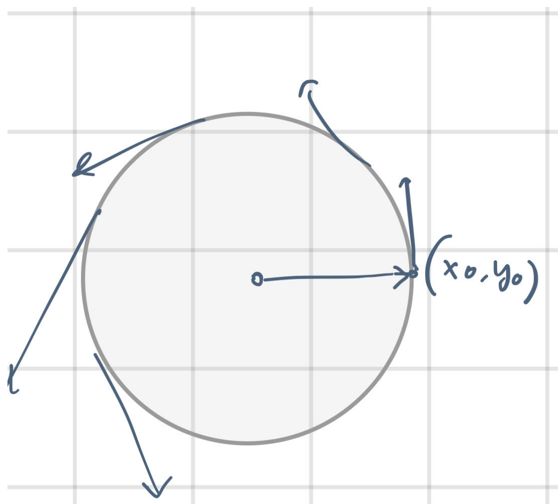



FIGURE 41. Integral Curve (circle!) of $X(x,y) = (-y, x)$

If $\gamma(0) = (0,0)$, then $\gamma(t) = (0,0)$ is the integral curve starting at $(0,0)$.
**Remark.** Let

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^2 &\to \mathbb{R}^2 \\ \Phi_t(x) := \Phi(t,x) &= R(t)x \end{aligned} \tag{3.28}$$

then $\Phi_{t+s} = \Phi_s \circ \Phi_t$ and $t \mapsto \Phi_t$ defines a group homomorphism $\mathbb{R} \to D \iff (\mathbb{R}^2)$. Note that $\gamma$ is defined everywhere on $\mathbb{R}$. Such vector fields with whole of $\mathbb{R}$ as their domains are called complete.

4. Consider $X(x) = x^2 \in \mathfrak{X}(\mathbb{R})$.
If $\gamma$ is an integral curve with $\gamma(0) = a \neq 0$ then,

$$\begin{aligned} \gamma'(t) = \gamma^2(t) \implies \frac{\gamma'(t)}{\gamma^2(t)} &= 1 \\ \int_0^t \frac{\gamma'(t)}{\gamma^2(t)} ds &= \int_0^t ds \\ \implies \gamma(t) &= \frac{a}{1-at}. \end{aligned}$$

The maximal domain for $\gamma$ is thus $\left(-\infty, \frac{1}{a}\right)$ for $a > 0$ and $\left(\frac{1}{a,\infty}\right)$ for $a < 0$. $X(0) = 0$, so $\gamma(t) = 0$ is the constant integral curve at 0.

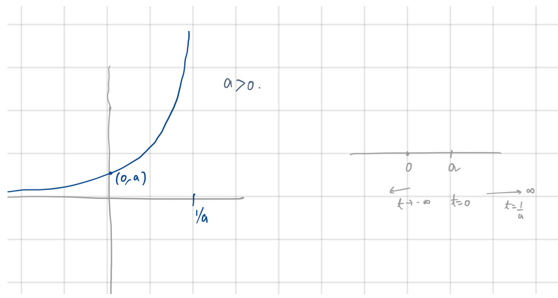



FIGURE 42. Integral Curve for $X(x) = x^2$. Not defined for all $t$.

5. Consider $M = \mathbb{R}^2 \setminus \{1,0\}$ and $X(x,y) = (1,0) \in \mathfrak{X}(M)$. The integral curve starting at $(0,0)$ will be $\gamma(t) = (t,0)$, $\gamma : (-\infty, t) \to M$.

This gives us a hint that the in-completeness of integral curve is not really due to a defect in vector field, but could be due to a defect of the manifold.

We will now ask the question of existence of integral curve of a vector field $X \in \mathfrak{X}(M)$ at $p$ in a smooth manifold $M$.

If $(U, x_1, \ldots, x_n)$ is a chart of $M$ cenetered at $p$. Then,

$$X|_U = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}; a_1, \ldots, a_n \in C^\infty(U).$$

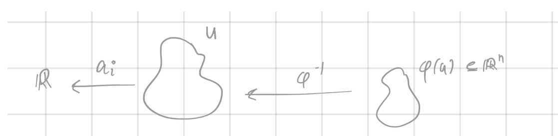



FIGURE 43

Define $b_i : \phi(U) \to \mathbb{R}$, $b_i = a_i \circ \phi^{-1} \in C^\infty(\phi(U))$.

Consider $Y = \phi_* X|_U$ ($\phi$ is a diffeomorphism) and the following curve $\gamma$ starting at 0.

$$\gamma'(t) = Y(\gamma(t)) = \left(b_1\left(\gamma(t)\right), \ldots, b_n(\gamma(t))\right), \ \gamma(0) = 0.$$

If this integral curve of $Y \in \mathfrak{X}(\phi(U))$ exists then $\phi^{-1} \circ \gamma$ is an integral curve of $X$ starting at $p$.

Let's check this:

**Theorem 3.26.** *Let $V \subseteq \mathbb{R}^n$ be an open set, $f : V \to \mathbb{R}^n$ a $C^\infty$ function, then the differential equation*

$$\frac{dy}{dt} = f(y), \; y(0) = p_0 \in V$$

*has a unique solution $y : J_{p_0} \to V$ where $J_{p_0}$ is an open interval in $\mathbb{R}$.*

The uniqueness means that, if $z : (a, b) \to V$ is also a solution i.e.

$$\frac{dz}{dt} = f(z), \; z(0) = p$$

then $(a, b) \subseteq J_{p_0}$ and $z(t) = y(t) \; \forall t \in (a, b)$.

Furthere there is a smooth dependence on the initial condition $p_0$.

**Theorem 3.27.** *Let $V \subseteq \mathbb{R}^n$ be open, $f : V \to \mathbb{R}^n$ a $C^\infty$ function. Then for each $p_0 \in V \; \exists \epsilon > 0$, a neighborhood $W \subseteq V$ of $p_0$, and $\Theta : (-\epsilon, \epsilon) \times W \to V$ also $C^\infty$ s.t.*

$$\frac{\partial \Theta}{\partial t}(t, p) = f\left(\Theta(t, p)\right) \;\; \text{and} \; \theta(0, p) = p \; \forall p \in W, t \in (-\epsilon, \epsilon.)$$

We can then extend these results to the smooth manifold $M$.

**Proposition 3.28.** *Let $M$ be a smooth manifold, and $X \in \mathfrak{X}(M)$. There $\exists$ an open set $J \subseteq \mathbb{R} \times M$ and a smooth function $\theta : J \mapsto M$ s.t.*

1. *$J_p := \{t \in \mathbb{R} | \, (t, p) \in J\}$ is an open interval containing $0$.*
2. *$\theta^{(p)} : J_p \to M, \; \theta^{(p)} = \theta\,(t, p)$. $\theta^{(p)}$ is the integral curve of $X$ starting at $p$ with maximal domain $J_p$. i.e. $\theta^{(p)}(0) = \theta(0, p) = p \; \forall p \in M$ and*

$$\frac{\partial \theta}{\partial t}\,(t, p) = X\left(\theta(t, p)\right) \;\; \forall \, (t, p) \in J. \tag{3.29}$$

*Remark* 3.29. $\theta$ is called the *flow* of the vector field $X$, and $J$ is called the flow domain.

Consider the following curves,

$$\begin{aligned} \gamma(t) &= \theta(t + s, p), \\ \delta(t) &= \theta(t, q), \; \text{where } q = \theta(s, p) \end{aligned} \tag{3.30}$$

Note that $\gamma(0) = q = \delta(0)$. And they satisfy,

$$\begin{aligned} \gamma'(t) &= \frac{\partial \theta}{\partial t}(t + s, p) = X\left(\theta\left(t + s, p\right)\right) \\ \delta'(t) &= X\left(\delta(t)\right) \end{aligned} \tag{3.31}$$

That is, both are integral curves of $X$ assing through the same starting point $q$. By uniqueness $\gamma(t) = \delta(t)$ wherever they are defined in the maximal set. We thus have the following nice property of the flow,

$$\theta(t+s,p) = \theta(t,\theta(s,p)) \tag{3.32}$$

or when written in the notation, $\theta_t(q) = \theta(t,q)$,

$$\theta_{t+s}(p) = \theta_t \circ \theta_s(p). \tag{3.33}$$

The intuition is clear, if we flow from a point $p$ for time $s$ and then flow from the new point for time $t$, it is same as flowing for a time $t+s$ from $p$.

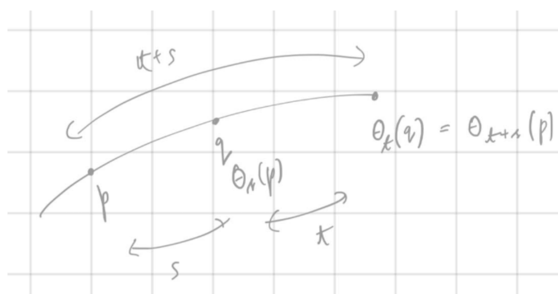



FIGURE 44

This immediately gives us the following lemma,

**Lemma 3.30.** $t+s \in J_p \iff t \in J_q$ *where* $q = \theta(s,p)$. *So,*

$$J_{\theta(s,p)} = J_p - s = \{t - s \mid t \in J_p\} \tag{3.34}$$

**Examples.**

1. Consider $X \in \mathfrak{X}(\mathbb{R}), X(x) = x^2$. We found the integral curves for this vector field aloing and their maximal domains to be,

$$\theta = \frac{a}{1-at}, \quad J_a = \begin{cases} \left(\frac{1}{a},\infty\right) & a<0 \\ \mathbb{R} & a=0 \\ \left(-\infty,\frac{1}{a}\right) & a>0 \end{cases} \tag{3.35}$$

Thus, $X$ has the following flow domain,

$$J = \{(x,a) \in \mathbb{R}\times\mathbb{R} \mid ax < 1\} \tag{3.36}$$

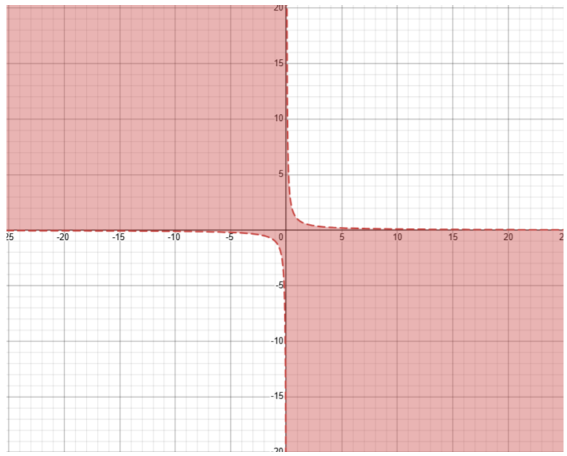


FIGURE 45

2. Consider $M = S^2, X \in \mathfrak{X}(S^2)$ such that $X(x,y,z) = (-y,x,0)$. The flow domain of this vector field is whole of $\mathbb{R} \times S^2$, where the flow is given by,

$$\theta(t,p) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} p, \quad p = (x_0, y_0, z_0) \tag{3.37}$$

$$= (x_0 \cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t, z_0) \tag{3.38}$$

This flow is just the rotation of $S^2$ around z-axis.

**Definition 3.31.** A vector field $X \in \mathfrak{X}(M)$ is called complete if the flow domain of $X$ is $\mathbb{R} \times M$.

Note that among the two examples discussed above one of the special property $S^2$ holds is compactness. Indeed on compact manifolds, all vectors fields are complete as we will see.

*Remark* 3.32. If $X \in \mathfrak{X}(M)$ is a complete vector field. Let $\theta_t : M \to M$ be defined by $\theta_t(p) = \theta(t,p)$. Note that,

- $\theta_0 = Id_M$.
- $\theta_{t+s} = \theta_t \circ \theta_s$
- $\theta_t = \theta_{-t}^{-1} \implies \theta$ is a diffeomorphism!

The map $t \mapsto \theta_t$ is a group homomorphism, $\mathbb{R} \to \text{Diff}(M)$. $\{\theta_t\}$ is called a one parameter group of diffeomorphisms of M.

**Lemma 3.33** (Uniform Time)**.** *Let $X \in \mathfrak{X}(M)$ and $J$ be the flow domain of $X$. If $\exists \epsilon > 0$ s.t. $(-\epsilon, \epsilon) \in J_p$ $\forall p \in M$, then $X$ is complete.*

*Proof.* The intution of the result seems clear. It basically follows from the nice property of flows 3.35. Suppose $X$ is not complete, then $\exists p \in M$ s.t. $J_p = (a, b) \neq \mathbb{R}$. Suppose $b < \infty$. Let $s \in (b - \epsilon, b)$, and let $q = \theta(s, p)$. $(-\epsilon, \epsilon) \subseteq J_q \implies (s - \epsilon, s + \epsilon) \subseteq J_p$ ( since $J_p = J_q + s$), which is a contradiction. Thus $b = \infty$. Similary, $a = \infty$. Thus $J = \mathbb{R}$ and $X$ is complete.

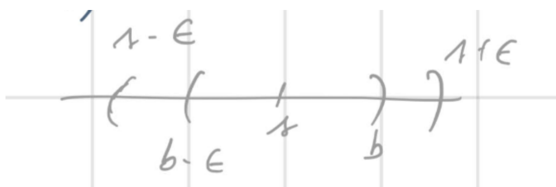



FIGURE 46

∎

Now to the result we promised.

**Theorem 3.34.** *$M$ a smooth manifold, and $X \in \mathfrak{X}$.*

(1) *If $X$ is compactly supported, then $X$ is complete.*
(2) *If $M$ is compact, then any $Y \in \mathfrak{X}(M)$ is complete.*

*Proof.* Let $K = \text{Support}(X) = closure\{p \in M \mid X(p) \neq 0\}$. If $p \in M - K, J_p = \mathbb{R}$ (constant vanishing vector field). For any $p \in M, \exists \epsilon(p) > 0$ and a neighborhood $U_p \subseteq M$ of $p$, s.t. $(-\epsilon(p), \epsilon(p)) \subseteq J_q$ $\forall q \in U_p$ i.e. $(-\epsilon(p), \epsilon(p)) \times U_p \subseteq J.K \subseteq \bigcup_{p \in K} U_p$. Since $K$ is compact,

$$\exists p_1, \ldots, p_n \; s.t. \; K \subseteq U_{p_1} \cup \ldots \cup U_{p_n}.$$

Now, let $\epsilon = min\{\epsilon(p_1), \ldots, \epsilon(p_n)\}$, then $(-\epsilon, \epsilon) \subseteq J_p$ for all $p \in M$. From the last lemma, $X$ is complete. If $M$ is compact, then $X$ has a compact support since any closed susbset of $M$ is compact. ∎

We will now tie some loose ends before going into the theory of integration on manifolds (and how it integrates with differentiation).

3.7. **Bump Functions. Goal.** If $M$ a smooth manifold, $p \in M, U \subseteq M$ is open s.t $p \in U$, then To get a function $f \in C^\infty(M)$ s.t. $supp(f) \subseteq U$, and $\exists V$ a neighborhood of $P$, $V \subseteq U$ s.t.

$$f|_V = 1, 0 \leq f(x) \leq 1.$$

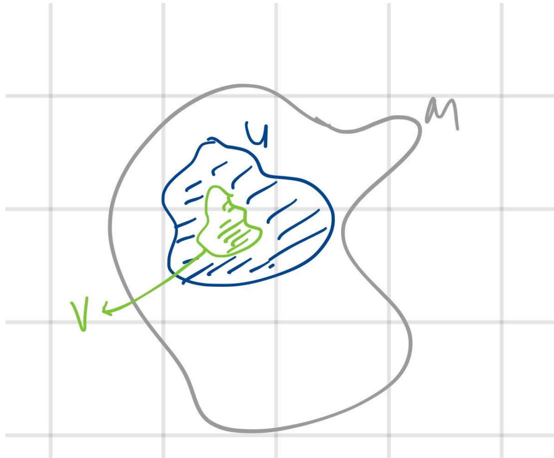



FIGURE 47. Enter Caption

To show such a function exists, we proceed in steps.

**Step-1: Bump Function on $\mathbb{R}$.** Given $r_1 < r_2, \exists h : \mathbb{R} \to \mathbb{R}$ such that $0 \leq h(x) \leq 1$ and

$$h(x) = \begin{cases} 1 & x \leq r_1 \\ 0 & x \geq r_2 \end{cases}$$

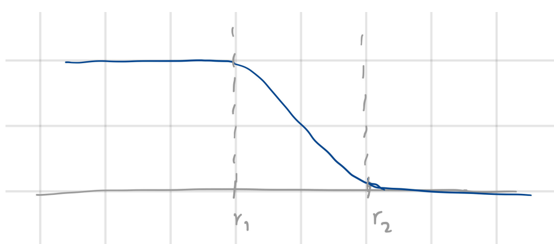



FIGURE 48. The function need not be decreasing in $[r_1, r_2]$.

One can get such a function via $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

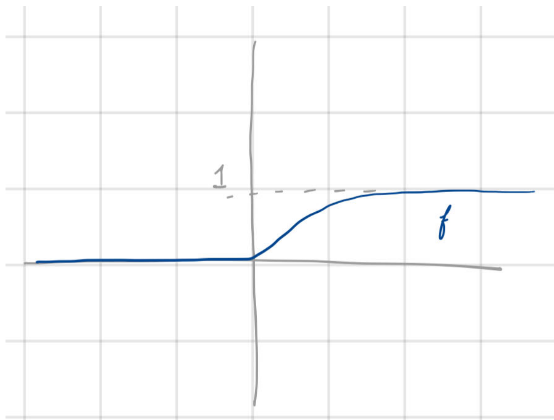



FIGURE 49

$f$ is smooth. First note that,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{e^{-1/h}}{h} = \lim_{t\to\infty} \frac{t}{e^t} = 0 = \lim_{h\to 0^-} = \frac{f(0+h)-f(0)}{h} = 0.$$

So, $f$ is indeed differentiable at 0 and

$$f(x) = \begin{cases} e^{-1/x} \cdot \frac{1}{x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

which is continuous at $x = 0$, by a similar proof above,

$$\lim_{h\to 0^+} \frac{e^{-1/h}}{1/h^2} = \lim_{t\to\infty} \frac{t^2}{e^t} \to \frac{2t}{e^t} \to \frac{2}{e^t} \to 0.$$

Then, we can proceed via induction to show $f$ is $C^k$ for all $k \in \mathbb{N}$. Thus, $f$ is smooth.

Now define, $h(x) = \frac{f(r_2-x)}{f(r_2-x)+f(x-r_1)}$. This function has the property we are looking for but as a function on $\mathbb{R}$ vi.z $f(x)|_{(\infty,0)} \equiv 1, (\infty, 0) \in supp(f) \subseteq \mathbb{R}, 0 \leq f(x) \leq 1$. It is then easy to extend this to $\mathbb{R}^n$ and to a smooth manifold $M$ via local charts.

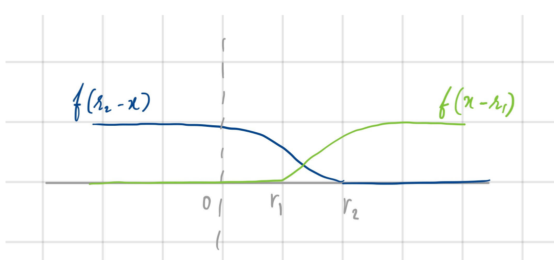



FIGURE 50. Bump function on $\mathbb{R}$.

**Remark.** We cannot find such functions on $\mathbb{C}$. Even real analytic functions are not possible to find. This is a consequence of the *Identity theorem* for analytic functions.

**Step-2: Bump Function on $\mathbb{R}^n$.** For $0 < r_1 < r_2, \exists H : \mathbb{R}^n \to \mathbb{R}$ smooth such that $H|_{B_{r_1}(0)} = 1,\ h(x) = 0\ \forall x \in \mathbb{R}^n \setminus B_{r_2}(0)$ and $0 \leq h(x) \leq 1\ \forall x \in \mathbb{R}^n$. This function can be obtained by using $h$, the bump function on $\mathbb{R}$ as, $H(x) = h(\|x\|)$.

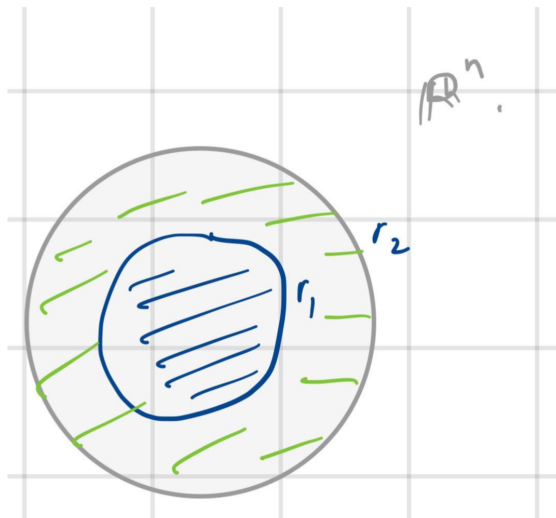



FIGURE 51. Bump Function on $\mathbb{R}^n$

**Step-3: Bump function on $M$.** Let's take $(U, \phi)$ a chart of $M$ centered at $p$, and $W \subseteq U$. Suppose $\overline{B_\epsilon(0)} \subseteq \phi(W)$. Then let $H : \mathbb{R}^n \to \mathbb{R}$ be defined as in step-2 with $r_1 = \epsilon/2,\ r_2 = \epsilon$.

Then, define $f : M \to \mathbb{R}$ by

$$f(x) = \begin{cases} H(\phi(x) & x \in W \\ 0 & x \notin W. \end{cases}$$

Let $V = \phi^{-1}(B_{\epsilon/2}(0))$. Then $f(x)|_V \equiv 1$. This is the required function!

**Can we extend a function defined on an open set in $M$ to the whole of $M$?**

No. This is not possible even in $\mathbb{R}$. For example, take $f(x) = 1/x$; there is no way we can extend it beyond $(0, \infty)$ smoothly to the whole of $\mathbb{R}^n$. However, we can extend it to the whole of $\mathbb{R}$ by defining a function that agrees with it only on a smaller open set.

**Proposition 3.35.** *Suppose $M$ is a smooth manifold, $p \in M$ and $U$ a neighborhood of $p$ and $f \in C^\infty(M)$, then $\exists \tilde{f} \in C^\infty(M)$ such that,*

$$\tilde{f}|_V = f|_V \text{ for some neighborhood } V \subseteq U \text{ containing } p.$$

We will see that this is true not just for a point but also for a closed set.

*Proof.* We can use the bump function $g \in C^\infty(M)$ satisfying , $g|V \equiv 1$ for some neighborhood $V \subset U$ of $p$ such that $supp(g) \subset U$ to define $\tilde{f} : M \to \mathbb{R}$,

$$\tilde{f}(x) = \begin{cases} f(x) \cdot g(x) & x \in U \\ 0 & x \notin U. \end{cases}$$

□

**Proposition 3.36.** *A vector field $X$ on $M$ is smooth if and only if for any $f \in C^\infty(M)$, $Xf$ is also smooth.*

*Proof.* $\implies$ : If $X$ is a smooth vector field and $(U, \phi)$ is a chart of $M$, then using the local frame we can write,

$$X = a_i \frac{\partial}{\partial x_i} \text{ where } a_i \in C^\infty(U).$$

So, $Xf|_U = a_i \frac{\partial f}{\partial x_i} \in C^\infty(U)$ since $a_i, f$ are smooth.

$\impliedby$ : It is enough to show that for any $p \in M. \exists V$ neighborhood of $p$ such that $X : V \to TV$ is smooth.

Let $(U, \phi)$ be a chart around $p$ in $M$. And let $X = a_i \frac{\partial}{\partial x_i}$; $a_i : U \to \mathbb{R}$ which may not be smooth. Consider $\tilde{\phi}_i \in C^\infty(M)$ extensions of $\phi_i \in C^\infty(U)$ such that

$$\tilde{\phi}_i|_{V_i} = \phi_i|_{V_i} \text{ for neighborhoods } V_i \subseteq U \text{ of } p.$$

Since, $Xf$ is smooth for all $f \in C^\infty(M)$ we have on $V = \bigcap_{i=1}^n V_i$,

$$X\tilde{\phi}_i|_V = a_i \text{ is smooth.}$$

Thus, $X|_V$ is smooth! □

So, charts (their projections onto $\mathbb{R}$) are primary examples of functions defined only locally. With the earlier proposition, we can smoothly extend these to $M$! Using this and the fact that smoothness is a local property, we were able to show the above nice proposition for the smoothness of vector fields. In fact, we can also similarly extend vector fields.

**Proposition 3.37.** *Suppose $M$ is a smooth manifold, $p \in M$, $U$ a neighborhood of $p$ and $X \in \mathfrak{X}(U)$, then $\exists\ \tilde{X} \in \mathfrak{X}(M)$ such that*

$$\tilde{X}|_V = X|_V \textit{ for some neighborhood } V \subseteq U \textit{ containing } p.$$

3.8. **Lie Bracket of Vector fields.** Consider on a smooth manifold $M$, two vector fields $X, Y \in \mathfrak{X}(M)$. Then $X(Yf) \in C^\infty(M)$ if $f \in C^\infty(M)$. $XY$, when defined this way, is not a derivation!

$$\begin{aligned}(XY)(fg) =& X(g(Yf) + f(Yg)) \\ =& gX(Yf) + (Yf)(Xg) + fX(Yg) + (Yg)(Xf) \\ (YX)(fg) =& Y(g(Xf) + f(Xg)) \\ =& gY(Xf) + (Xf)(Yg) + fY(Xg) + (Xg)(Yf)\end{aligned}$$

But when these two are subtracted, we see that,

$$(XY - YX)(fg) = f(XY - YX)(g) + g(XY - YX)f.$$

So, $(XY - YX)$ is a linear map satisfying the Leibnitz rule. We thus have the following proposition.

**Proposition 3.38.** *Let $M$ be a smooth manifold and $X, Y \in \mathfrak{X}(M)$, then we can define the vector field $[X, Y]$ called the Lie Bracket of vectors fields $X, Y$ by,*

$$[X, Y](p)f_p = ([X, Y]f)(p) = X(p)(Yf)_p - Y(p)(Xf)_p$$

*for any $p \in M$ & $f_p, g_p \in C_p^\infty(M)$. Here $f, g$ are the extensions of $f_p, g_p$ on $M$.*

**Remarks.**
1. This is only possible because the map $C^\infty(M) \to C_p^\infty(M)$ is a surjection (due to the existence of the bump function). We define a vector field using a derivation on $C^\infty(M)$. This illustrates the isomorphism between $\mathfrak{X}(M)$ and $Der(C^\infty(M))$ which we didn't prove earlier. So, this doesn't hold directly for complex manifolds.
2. $[X, Y]$ is smooth since for any function $f$, $[X, Y]f = X(Yf) - Y(Xf)$ is smooth.
3. $[\ .,.] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfies following properties
   (1) *Anti-Commutativity*: $[X, Y] = -[Y, X]$.
   (2) *Bi-Linearity*

$$\begin{aligned}[X, aY + bZ] =& a[X, Y] + b[Y, Z] \\ [aX + bY, Z] =& a[X, Z] + b[Y, Z]\end{aligned}$$

   (3) *Jacobi Identity*:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

   or equivalently,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The first form of Jacobi identity looks like the Leibnitz rule, and in fact, this is related to the Lie-Derivative.

**Definition 3.39.** Let $V$ be a $K$-vector space equipped with binary operation $[.,.] : V \times V \to V$, which is anti-commutative, bilinear and satisfies Jacobi identity, then V is called a Lie Algebra over $K$.

So, $\mathfrak{X}(M)$ is an infinite-dimensional Lie Algebra over $\mathbb{R}$.

**Question.** How do we connect this to the Lie Algebra defined by tangent space at the identity of a Lie Group? Here's a rough sketch.

Let $G$ be a Lie Group and $\mathfrak{g} = T_eG$. If $v \in T_eG$, and $L_g : G \to G,\ L_g(h) = gh$ a diffeomorphism.

**Goal.** To define a binary operation on $\mathfrak{g}$ with the Lie Algebra structure.

We can define *all* the vector fields on $G$ through, $X_v(g) = L_{g*}(v) \in T_gG$. So we have a map $\mathfrak{g} \to \mathfrak{X}(M)$. We can then define the operation

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

by letting $[v, w]$ map to $X_{[v,w]} = [X_v, X_w]$! Basically, we are claiming a bi-linear homomorphism between $[,]$ on $\mathfrak{X}(M) \times \mathfrak{X}(M)$ and $[,]$ on $\mathfrak{g} \times \mathfrak{g}$.

We can find the local form of Lie Bracket of vector fields.

**Proposition 3.40.** *Suppose $(U, x_1, \ldots, x_n)$ is a chart of $M$ and*

$$X|_U = \sum X_i \frac{\partial}{\partial x_i}, \quad X_i \in C^\infty(M)$$
$$Y|_U = \sum Y_i \frac{\partial}{\partial x_i}, \quad Y_i \in C^\infty(M)$$

*then,*

$$[X,Y]\big|_U = \sum_{i,j} \left( X_i \frac{\partial Y_i}{\partial x_i} - Y_i \frac{\partial X_i}{\partial x_i} \right) \frac{\partial}{\partial x_j} = \sum_j (XY_j - YX_j) \frac{\partial}{x_j}.$$

*Proof.* Let's use some physicsy notation. $\frac{\partial}{\partial x_i} \equiv \partial_i$ and summation over repeated indices.

$$\begin{aligned}
[X,Y]f =& X(Yf) - Y(Xf) \\
=& X(Y_j \partial_j f) - Y(X_j \partial_j f) \\
=& (X_i \partial_i Y_j \partial_j f + X_i Y_j \partial_i \partial_j f) - (Y_i \partial_i X_j \partial_j f + Y_i X_j \partial_i \partial_j f) \\
=& (X_i \partial_i Y_j - Y_i \partial_i X_j)\, \partial_j f
\end{aligned}$$

$\square$

**Examples.**

1. $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > 0\}$.

Take the following vector fields in $\mathfrak{X}(M)$,

$$X = \frac{\partial}{\partial x_i} - \frac{x_2}{x_1^2 + x_2^2}\frac{\partial}{\partial x_3},$$
$$Y = \frac{\partial}{\partial x_2} + \frac{x_1}{x_1^2 + x_2^2}\frac{\partial}{\partial x_3}$$

Let's write them in the local form. We have a single chart here, so things are very straightforward.

$$[X, Y] = (XY_j - YX_j)\frac{\partial}{\partial x_j}$$
$$= (XY_3 - YX_3)\frac{\partial}{\partial x_j}$$

because constant functions have a vanishing derivation.

So,

$$XY_3 = \frac{\partial}{\partial x_1}(\frac{x_1}{x_1^2 + x_2^2}) = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$$
$$YX_3 = \frac{\partial}{\partial x_2}(\frac{-x_2}{x_1^2 + x_2^2}) = \frac{x_2^2 - x_1^2}{x_1^2 + x_2^2}$$

$[X, Y] = 0$!

3.8.1. *Comments on Geometric Interpretation of Lie Brackets.* The Lie Brackets describe in some sense to what extent the mixed derivatives (fail to) commute, and geometrically, this translates to (failure of) commutation of flows along the two vector fields.

**Definition 3.41.** Let $X, Y \in \mathfrak{X}(M)$ with flows $\Theta, \Phi$. The flows are called commuting if $\forall p \in M$ and open intervals $J, K$ containing 0 either $\Theta_t(\Phi_s(p))$ or $\Phi_s(\Theta_t(p))$ is defined for all $(s, t) \in J \times K$ then both are defined and are equal.

**Theorem 3.42.** $[X, Y] = 0$ *is and only if the flows of* $X$ *and* $Y$ *commute.*

An important application of Lie Brackets is the Frobenius' Theorem.

[Potential Excursion - Lie Derivative]

We now move on to the integration sector of differential geometry. As we will soon see, the integration theory brings the boundary of a manifold to the picture, so we will first start with a discussion on manifolds with boundaries.

## 4. Manifold with Boundary

**Definition 4.1.** A topological manifold with boundary of dimension $n$ is a Hausdorff, 2nd countable topological space $M$ such that for any point $p \in M$, there is a neighbourhood $U$ homeomorphic to an open set of $\mathbb{H}^n$.

We denote, $\mathbb{H}^{\ltimes} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and

$$\text{Int } \mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}, \ \partial\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}.$$

As usual, if $U \subseteq M$ is homeomorphic to $V \subseteq \mathbb{H}^n$ open and $\phi : U \to V$ is the homeomorphism, then $(U, \phi)$ is called a chart of M. Let's see some examples of such manifolds. These follow from a theorem we will prove shortly.

**Examples.**

1. $\mathbb{H}^n$ is an n-manifold with boundary with charts given by

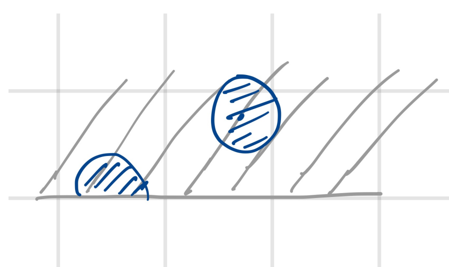


Figure 52

2. The closed ball $\overline{B^n} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

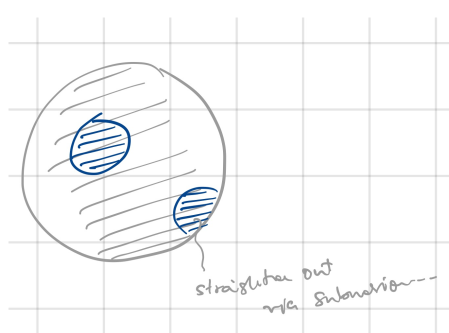



Figure 53

3. The hemisphere $U \subset S^n, U = \{x \in S^n \mid x_{n+1} \geq 0\}$.

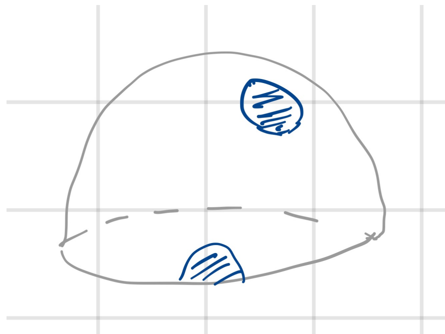


FIGURE 54

If $M$ is an n-manifold with boundary:

- $x \in M$ is called an interior point if there is a chart $U, \phi$ of M such that $\phi(x) \in Int\ \mathbb{H}^n$. $Int\ \mathbb{H}^n$ is the set of all interior points of $M$. Infact since $Int\ \mathbb{H}^n$ is open, $Int\ \mathbb{M} \subset M$ is open.
$$Int(M) = \{x \in M \mid \exists (U, \phi) \text{ chart of } M\ s.t.\ \phi(x) \in int\ (\mathbb{H}^n)\}$$
- $x \in M$ is called a boundary point if for some chart $(U, \phi)$ of $M$, $\phi(x) \in \partial\mathbb{H}^n$.
$$\partial M = \{x \in M \mid \exists (U, \phi) \text{ chart of } M\ s.t.\ \phi(x) \in \partial\mathbb{H}^n\}$$
is called the boundary of $M$.

Can one then have two charts containing a point, one taking it to the interior of $\mathbb{H}^n$ and the other to the boundary of $\mathbb{H}^n$? Thankfully no!

**Theorem 4.2.** $Int(M) \cap \partial M = \emptyset$.

*Proof.* Uses local homology groups. However, we can prove this easily for *smooth* manifolds. ☐

**Corollary 4.3.** $\partial M = M \setminus Int(M)$ *is closed.*

4.1. **Smooth structure on topological manifolds with boundary.** The subtlety here is in defining smoothness on sets not necessarily open in $\mathbb{R}^n$! We can do this by demanding a smooth extension of such a function on a neighbourhood in $\mathbb{R}^n$.

**Definition 4.4.** Let $S \subseteq \mathbb{R}^n$ be any set (not necessarily open) and $f : S \to \mathbb{R}^n$. Then $f$ is called smooth if for any $x \in S$ there is a neighborhood $U \subseteq \mathbb{R}^n$ of x and a smooth function $\tilde{f} : U \to \mathbb{R}^n$ such that,
$$\tilde{f}\big|_{S\cap U} = f\big|_{S\cap U}.$$

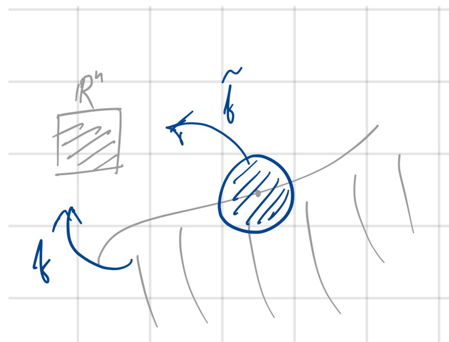


FIGURE 55

$U$ can be chosen as an open ball. Of course, this is trivial and useless for a discrete space $S$.

**Definition 4.5.** A smooth atlas on an n-manifold with boundary $M$ is a collection of charts $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ such that,

$$\phi_\alpha \circ \phi_b^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is smooth $\forall \alpha, \beta \in A$.

**Definition 4.6.** A smooth manifold with a boundary is a topological manifold with a boundary with a choice of a maximal smooth atlas

Now, to the theorem we promised.

**Theorem 4.7.** *Let $M$ be a smooth n-manifold with boundary. If $(U, \phi)$ is a chart of $M$ such that $\phi(x) \in \partial\mathbb{H}^n$ for some $x \in U$, then for any other chart $(V, \psi)$ with $x \in V, \psi(x) \in \partial\mathbb{H}^n$.*

**Remark.** There is a subtlety here. We need to argue that the topological manifold with a boundary is the same as the smooth manifold with a boundary. We will skip this, assuming the theorem - (3.1).

*Proof.* Let's $x \in M$ and $\phi(x) = 0$ (this is possible WLOG only on the boundary). Suppose $\exists(V, \psi)$ such that $\psi(x) \in Int(\mathbb{H})^\ltimes$.

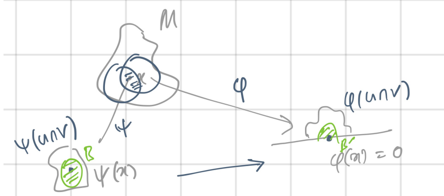


FIGURE 56

Choose $B = B_\delta(\psi(x)) \subseteq \psi(U \cap V)$ such that $B \subseteq Int(\mathbb{H})^n$. Now $f(y) := \phi \circ \psi^{-1}(y) = 0$, where $y = \psi(x)$. Consider

$$g : \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V).$$

By smooth compatibility, $\exists B' = B_\epsilon(0)$ and $\tilde{g} : B' \to \mathbb{R}^n$ such that

$$\tilde{g} \mid_{B' \cap \phi(U \cap V)} = g \mid_{B' \cap \phi(U \cap V)} .$$

We can shrink $B$ if needed (taking smaller $\delta$ by continuity) such that $f(B) \subseteq B' \cap \phi(U \cap V)$.

Then $\tilde{g} \circ f = Id_B$. This means $J_f(y')$ is invertible for any $y' \in B$. So, there are open neighborhoods $U_{y'} \subseteq B$ of $y'$ and $V_{f(y')} \subseteq B'$ of $f(y')$ such that $f : U_{y'} \to V_{f(y')}$ is a diffeomorphism. $V_{f(y')}$ is open in $\mathbb{R}^n$ since $B'$ is open in $\mathbb{R}^n$.

By local criterion for openness, $f(B)$ is open in $\mathbb{R}^n$. But $f(B) \subseteq \mathbb{H}^n$ and $0 \in f(B)$ which gives a contradiction. $\square$

The above is basically a proof for the following small result.

**Lemma 4.8.** *Let $U \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^n$ be an arbitrary subset and $f : U \to S$ a diffeomorphism. Then $S$ is open in $\mathbb{R}^n$.*

Here's a nice way to get hold of manifolds with boundaries.

**Theorem 4.9.** *Let $M$ be a smooth manifold and $f \in C^\infty(M)$. If $a \in \mathbb{R}$ is a regular value of $f$ then,*

$$M_a = \{x \in M \mid f(x) \leq a\}$$

*is a manifold with boundary where, $\partial M_a = f^{-1}(a)$ and $Int(M_a) = f^{-1}((-\infty, a))$.*

*Proof.* $M_{<a} = \{x \in M \mid f(x) < a\} \subset M$ is an open submanifold of $M$. If $p \in M_{<a}$, there exists chart $(U, \phi)$ around $p$ such that $U \subseteq M_{<a}$ (by a suitable translation or shrinking) such that

$$\phi : U \to \phi(U) \subseteq \mathbb{H}^n, \quad \phi(U) \cap \partial\mathbb{H}^n = \emptyset.$$

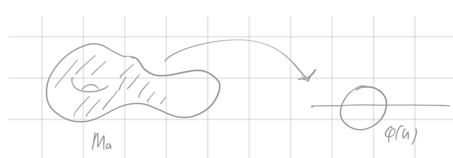



FIGURE 57

That is all points in $M_{<a}$ are mapped to $Int(\mathbb{H}^n)$.

If $p \in f^{-1}(a)$, then $f_* : T_pM \to T_a\mathbb{R} = \mathbb{R}$ is surjective. Let $g = a - f, g \in C^\infty(M)$. Then 0 is a regular value of $g$.

By submersion theorem, $\exists$ a chart $(V, \psi)$ around $p$ such that for any $(x_1, \ldots, x_n)$ in $\psi(V)$,

$$g \circ \psi^{-1}(x_1, \ldots, x_n) = x_n.$$

And $V_a = V \cap M_a = \{x \in V \mid x_n = g \circ \psi^{-1}(x) \geq 0\}$. This means $\psi(V_a) \subseteq \mathbb{H}^n$! And in particular $\psi(p) \in \partial \mathbb{H}^n$. We have thus provided charts on $M_a$ which precisely take $M_{<a}$ to the interior of $\mathbb{H}^n$ and $M_{=a}$ to the boundary of $\mathbb{H}^n$! $\square$

**Examples.**

1. $S = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1\}$.

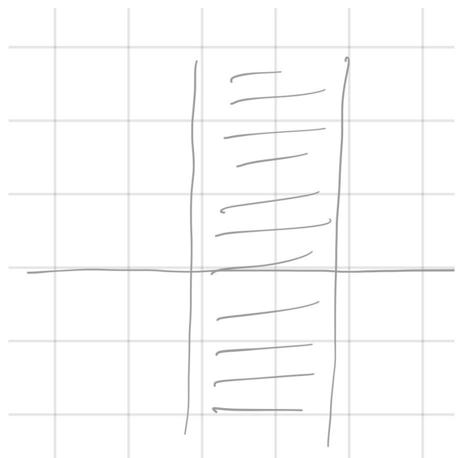


FIGURE 58

S is a 2-manifold with a boundary due to the function,

$$f : \mathbb{R}^2 \to \mathbb{R}, \; f(x, y) = x^2$$

with the regular value 1.

One can also deal with multiple functions to get hold of a manifold with a boundary. (Example postponed to problem set).

2. Torus $\mathbb{T}^2$ [Manifolds à la Morse]

$$\begin{aligned} &f : \mathbb{R}^2 \to \mathbb{R}^3 \\ &f(s, t) = ((2 + \cos t) \cos s, (2 + \cos t) \sin t, \sin t) \end{aligned}$$

$X = f(\mathbb{R}^2)$ is a compact submanifold of $\mathbb{R}^2$ which is diffeomorphic to the torus $\mathbb{T}^2$. Take the following function on $X$.

$$h : X \to \mathbb{R}, \; h(x, y, z) = x.$$

Then, any $a \in (-3, 3) \setminus \{\pm 1\}$ is a regular value of $h$ (as seen in Quiz I). Then the manifolds with boundaries we get are of form $h^{-1}((-\infty, a])$ with different values of $a$,

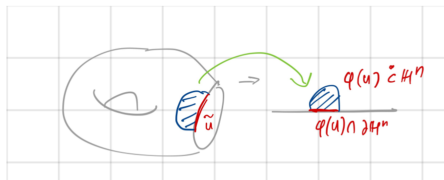



FIGURE 59

Such functions that give out information about the topology of manifolds as we move between the regular values (or critical points) are called Morse functions.

**Remarks.**

1. Any manifold with boundary $M$ such that $\partial M = \emptyset$ is a manifold.
2. If $M$ is an n-manifold with boundary then $Int(M)$ is an n-manifold. Because the charts map into open sets of $Int(\mathbb{H}^n)$ which are open in $\mathbb{R}^n$ too.

**Proposition 4.10.** *Let $M$ be an n-manifold with boundary, then $\partial M$ is a manifold of dimension $n-1$.*

This is precisely because $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$!

*Proof.* For any $p \in \partial M$, there is a chart $(U, \phi)$ of $M$, such that $\phi(U) \subseteq \mathbb{H}^n$ is open and $\phi(p) \in \partial\mathbb{H}^n$.

Let $\tilde{U} = U \cap \partial M$. Define

$$\tilde{\phi} : \tilde{U} \to \mathbb{R}^{n-1},\ \tilde{\phi}(q) = (\phi_1(q), \ldots, \phi_{n-1}(q)).$$

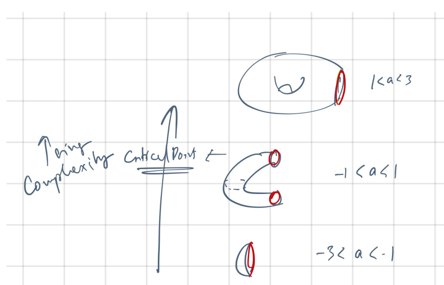



FIGURE 60

Then, $(\tilde{U}, \tilde{\phi})$ is a chart of $\partial M$. Smooth compatibility between intersecting charts trivially follows from the smoothness of $M$. □

Now to the delicate part of manifolds with boundaries. The tangent space of points in $\partial M$, is it an n or (n-1) dimensional vector space? Or is it something like a (upper) half of vector space? The convention is to work with n-dimensional vector spaces...

**Definition 4.11.** Let $M, N$ be smooth manifolds with boundary. A map $f : M \to N$ is smooth if for any chart $(V, \psi)$ of $N$, there is a chart $(U, \phi)$ of $M$ such that $f(U) \subset V$ and

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \text{ is smooth.}$$

The definition is exactly the same as that of manifolds, but of course, there is subtlety in asking the inter-manifold transition function to be smooth. That is, If the charts contain boundary points, then the smoothness is by earlier definition - where it is possible to extend the function smoothly to a neighbourhood in $\mathbb{R}^n$.

This way, we can define smooth functions from $M$ to $\mathbb{R}$ to take derivations. Let $C^\infty(M)$ be the set of smooth functions from $M$ to $\mathbb{R}$. And as usual, $C_p^\infty(M)$ is the ring of germs of smooth functions at $p$. That is,

$$C_p^\infty(M) = \{(U, f) \mid U \overset{open}{\subset} M, p \in U, f \in C^\infty(U)\}/\sim .$$

Recall, $(U, f) \sim (V, g) \iff \exists W \subset U \cap V$ neighborhoof of $p$, such that $f|_W = g|_W$.

**Definition 4.12.** The tangent space of a manifold with boundary $M$ at $p$ is defined as,

$$T_pM = Der(C_p^\infty(M)).$$

**An aside on gluing two manifolds with boundary.**

1. Let $M, N$ be manifolds with boundary and $f : M \to N$ a diffeomorphism. Then,

$$K = \frac{M \sqcup N}{\sim}, \quad q \sim f(q) \ \forall q \in \partial M$$

is a manifold without boundary.

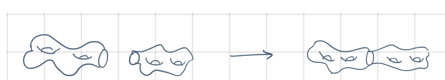


FIGURE 61

2. Visual boundary by definiong geodesic end-points. . . Compactification of $\mathbb{R}^n$. . .

Coming back to tangent spaces,

**Proposition 4.13.** *$dimT_pM = dimM$.*

*Proof.* If $U \subseteq M$ is open, $T_pU = T_pM$ because the local rings are same on $U$ and $M$.

Note that if $f : M \to N$ is a diffeomorphism,

$$f_* : T_pM \longrightarrow T_{f(p)}M, \quad f_*(v)(g_{f(p)}) = v\left((g \circ f)_p\right)$$

is an isomorphism.

For $p \in Int(M)$, it is very clear. $T_pM = T_p\left(Int(M)\right)$. So,

$$dimM = n \implies dimT_p\left(Int(M)\right) = n.$$

.

For $p \in \partial M$, the strategy is to first show that,

**Proposition 4.14.** *If $p \in \partial\mathbb{H}^n$, $i_*T_p\mathbb{H}^n \to T_p\mathbb{R}^n$ is an isomorphism. Where $i : \mathbb{H}^n \to \mathbb{R}^n$ is the (smooth) inclusion map.*


FIGURE 62

*Proof.* Note the pullback,

$$\begin{aligned} i^* &: C_p^\infty(\mathbb{R}^n) \to C_p^(\mathbb{H}^n) \\ &i^*[U, f] = [U \cap \mathbb{H}^n, f|_{U\cap\mathbb{H}^n}] = [i^{-1}(U), f \circ i]. \end{aligned}$$

This is a surjection, because by defintion any smooth function on an open set in $\mathbb{H}^n$ containing $p$ can be extended to a smooth function on an open set of $\mathbb{R}^n$.

Consider $[V, g] \in C_p^\infty(\mathbb{H}^n)$ then g can be extended to $[\tilde{V}, \tilde{g}] \in C_p^\infty(\mathbb{R}^n)$ such that $(V \cap \tilde{V}, \tilde{g}|_{V\cap\tilde{V}}) = \left(i^{-1}(\tilde{V}), \tilde{g} \circ i\right) \sim (V, g)$.

This extension need not be unique so $i^*$ is not an injection.

- $i_*$ is an injection:
  Suppose $i_*(v) = 0$ for some $v \in T_p\mathbb{H}^n$ . Then for any $f_p \in C_p^\infty(\mathbb{H}^n)$,

$$\begin{aligned} v(f_p) &= v(i^*\tilde{f}_p), \quad \tilde{f}_p \in C_p^\infty(\mathbb{R}^n) \\ &= i_*(v)\left(\tilde{f}_p\right) = 0. \end{aligned}$$

  So, $v = 0$.
- $i_*$ is a surjection:
  Let $w = w_i\frac{\partial}{\partial x_i} \in T_p\mathbb{R}^n$. Then define $v \in T_p\mathbb{H}^n$ by,

$$v(f_p) = w(\tilde{f}_p), \quad i^*\tilde{f}_p = f_p$$
$$= w_i \frac{\partial \tilde{f}}{\partial x_i}|_p.$$

This is well defined! Suppose $i^*\tilde{f}_p = i^*\tilde{g}_p = f_p$ and

$$\tilde{f}_p = [\left(U_1, \tilde{f}\right)], \tilde{g}_p = [(U_2, \tilde{g})]$$

So, there exists $W \overset{open}{\subset} U_1 \cap U_2 \cap \mathbb{H}^n$ and $p \in W$ s.t,

$$\tilde{f}|_W = \tilde{g}|_w = f|_W \implies \frac{\partial \tilde{f}}{\partial x_i}|_p = \frac{\partial \tilde{g}}{\partial x_i}|_p.$$

From the defintion it immediately follows that $v(f_p)$ is a a derivation, i.e. linear and obeys Leibnizarity.

$\square$

Now we can extend it to the manifolds (with boundary). Let $p \in \partial M$. $\exists (U, \phi)$ chart around $p$ such that $\phi(p) \in \partial \mathbb{H}^n$.

So,

$$\phi : U \to \phi(U) \subseteq \mathbb{H}^n \text{ is a diffeomorphism,}$$
$$\phi_* : T_pU \to T_{\phi(p)}\mathbb{H}^n \text{ is an isomorphism.}$$

Thus, $T_pU$ is spanned by $\{\phi_*^{-1}\left(\frac{\partial}{\partial x_i}\right) \mid i = 1, \dots, n\}$.

$\square$

We can also disucss vector bundles over manifolds with boundaries. They all hold similar to manifolds but we will postpone the dicussion to when we need them.

## 5. Differential Forms

5.1. **Co-tangent Bundle.** This is a special case of dual construction of a vector bundle.

If $M$ is a smooth manifold, we define

$$T_p^*M = Hom_{\mathbb{R}}\left(T_pM, \mathbb{R}\right)$$

as the cotangent space of $M$ at $p$. And

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

as the cotangent bundle of $M$. We will soon prove the topology, manifold and vector bundle structure of this set.

**Definition 5.1.** A 1-form is a function (subsection) ,

$$w : M \to T^*M$$

such that $w(p) \in T_p^*M$.

Elements of $T_p^*M$ are called cotangent vectors or covectors. And 1-form $w$ is also called a covector field.

**Example.** [A simple way to get hold of a 1-form]
If $f \in C^\infty(M)$, it defines a linear map,

$$(df)_p : T_pM \longrightarrow \mathbb{R}$$
$$v \longmapsto (df)_p (v) = v(f_p) = v(f).$$

Then $p \mapsto df|_p$ is a 1-form which we denote by $df$.

Infact, $df$ is nothing but $f_*$ after noting that $T_{f(p)}\mathbb{R} \cong \mathbb{R}$ because in this case,

$$f_*(v) = v(f)\frac{d}{dt}|_{f(p)} \to df(v)$$

In particular, if $(U, x_1, \ldots, x_n)$ is a chart of $M$, then $dx_1, \ldots, dx_n$ are 1-forms on $U$.

**Proposition 5.2.** *$dx_1|_p, \ldots, dx_n|_p$ is the dual basis of $\frac{\partial}{\partial x_i}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$.*

*Proof.* Indeed from the definition of $dx_i|_p$,

$$dx_i|_p\left(\frac{\partial}{\partial x_j}|_p\right) = \frac{\partial x_i}{\partial x_j}p = \delta_{ij}.$$

which form the basis for the dual space $T_p^*M$. □

Note that,

$$df|_p = a_i dx_i|_p, \text{ where, } a_j = df|_p\left(\frac{\partial}{\partial x_j}|_p\right) = \frac{\partial f}{\partial x_j}(p).$$

So, $df|_p = \frac{\partial f}{\partial x_i}(p)dx_i|_p$ or,

$$df = \frac{\partial f}{\partial x_i}dx_i$$

which looks very familiar from the calculus on Euclidean spaces!

From above, it's easy to derive the change of basis matrix for the coordinate 1-forms. Let's say there are two intersecting charts $(U, x_1, \ldots, x_n), (V, y_1, \ldots, y_n)$ with $p \in U \cap V$, then

$$dy_j = \frac{\partial y_j}{\partial x_i}dx_i.$$

The change of basis matric from $\{dx_i\}$ to $\{dy_j\}$ is given by,

$$\left(J_{\psi\circ\phi^{-1}}(x)\right)^t.$$

**Cotangent Bundle as a vector bundle.**

The projection map onto $M$ is,

$$\pi : T^*M \to M, \quad \pi(\alpha) = p, \text{ where } \alpha \in T_p^*M.$$

For any chart $(U, \phi)$ of $M$ with $\phi = x_1, \ldots, x_n$.

Let $T^*U = \bigsqcup_{p\in U} T_p^*M$. Then just as the case for $TM$, define

$$\tilde{\phi} : T^*U \to \phi(U) \times \mathbb{R}^n, \; \tilde{\phi}\left(c_i dx_i|_p\right) = \left(\phi(p), c_1, \ldots, c_n\right).$$

We give $T^*U$, the topology such that $\phi$ is a homeomorphism. (This gives a unique topology on $T^*U$).

Let $\mathcal{B} = \{W \subseteq T^*M \mid W \subseteq T^*U \text{ open for some chart } (U, \phi) \text{ of } M\}$. This forms a basis for a topology on $T^*M$.

We take the topology on $T^*M$ generated by $\mathcal{B}$. This satisfies,

$$W \subset T^*M \text{ open} \iff W \cap T^*U \text{ is open for all charts } (U, \phi) \text{ of } M.$$

Fill the comments on basis and topology. Prove that $\mathcal{B}$ indeed formas a basis...

So, $T^*M$ is a topological manifold of dimension $2n$ with charts of type $\left(T^*U, \tilde{\phi}\right)$, where $(U, \phi)$ is a chart of $M$.

These charts are also smoothly compatible. If $(U, \phi)$, $(V, \psi)$ are two charts of $M$, then

$$\begin{aligned}\tilde{\psi} \circ \tilde{\phi}^{-1} : \tilde{\phi}\left(T^*U \cap T^*V\right) &\longrightarrow \tilde{\psi}\left(T^*U \cap T^*V\right)\\ (p, c) &\mapsto \left(\tilde{\psi} \circ \tilde{\phi}(p), \left(\left(J_{\tilde{\psi}\circ\tilde{\phi}}\right)^{-1}\right)^t (p)c\right).\end{aligned}$$

We thus have, $(T^*M, M, \pi)$ as a smooth vector bundle of rank $n$ over $M$.

If $(U, x_1, \ldots, x_n)$ is a chart of $M$, then the coordinate 1-forms $dx_1, \ldots, dx_n$ are smooth subsections $dx_i : U \to T^*U = \pi^{-1}(U)$.

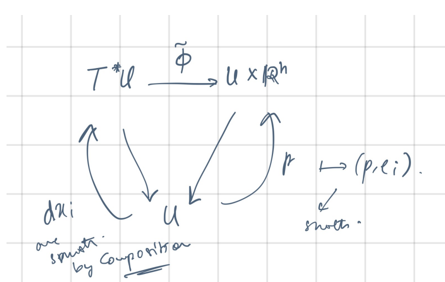



FIGURE 63

Thus, they form a smooth local frame of $T^*M$ over U.

**Proposition 5.3.** *If $w : U \to T^*U$ is a subsection then $w$ is smooth $\iff$ $\exists a_1, \dots, a_n \in C^\infty(U)$ such that $w = aidx_i$.*

We have proved this for any subsection of a vector bundel earlier.

**Proposition 5.4.** *Let $w$ be a 1-form on $M$. Then, the following are equivalent.*

(1) *$w$ is smooth.*
(2) *FOr any chart $(U, x_1, \dots, x_n)$, $\exists a_1, \dots, a_n \in C^\infty(U)$ such that $w = a_i dx_i$.*
(3) *For any $p$, there is a chart $(U, \phi)$ containing $p$ for which $\exists a_1, \dots, a_n \in C^\infty(M)$ such that $w = a_i dx_i$.*

The importance of 3 is that we just need check for charts that cover $M$ rather than the entire atlas. This follows from the first problem of PSet-2. It is enough to check smoothness for an atlas of $M$ contained in a maximal smooth atlas of M.

We denote the set of 1-forms on M by $\Omega^1(M)$.

$$\Omega^1(M) = \{w : M \to T^*M \mid w \text{ is smooth}\}.$$

**An aside.** Any vector space is isomorphic to its dual. So, it turns out $TM \cong T^*M$ even as vector bundles.

If $w$ is a 1-form on $M$, and $X$ is a vector field then we can define a function (like $Xf$ for $f \in C^\infty(M)$)

$$w(X) : M \to \mathbb{R}, \quad w(X)(p) = w_p(X_p)$$

where $w \in T_p^*M, X_p \in T_pM$.

Note that, if $g \in C^\infty(M)$ then $w(gX) = gw(X)$.

**Proposition 5.5.** *Let $w$ be a 1-form on $M$. $w$ is smooth $\iff$ $\forall X \in \mathfrak{X}(M)$ $w(X)$ is smooth.*

*Proof.* $\implies$ : If $(U, x_1, \dots, x_n)$ is a chart of $M$ and $w = a_j dx_j$ for $a_j \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$, then $X|_U = b_i \frac{\partial}{\partial x_i}$. So,

$$\begin{aligned} w(X)|_U &= a_j b_i dx_j \left( \frac{\partial}{\partial x_i} \right) \\ &= a_i b_i \in C^\infty(M). \end{aligned}$$

$\impliedby$ : Let $w$ be any 1-form. WE can write $w = a_i dx_i$, where $a_i$ need not be smooth. For $p \in M$, let $(U, x_1, \dots, x_n)$ be a chart of $M$ containing p. Our strategy is to extend the local vector fields $\frac{\partial}{\partial x_i}|_U$ on $U$ to whole of $M$.

Let $g$ be a bump function at $p$, $g : M \to \mathbb{R}$ smooth with $supp(g) \subseteq U$. Then $\exists$ neighborhood $V \subseteq U$ of $p$ such that $g|_V = 1$.

Let the vector field extension be defined as follows,

$$X_i(q) = \begin{cases} g(q)\frac{\partial}{\partial x_i} & q \in U \\ 0 & q \notin U. \end{cases} \tag{5.1}$$

Then, by the hypothesis $w(X_i)|_V = a_i|_V$ is smooth! So $w|_V$ is smooth. And since smoothness is a local property, $w$ is smooth. □

**Pull back of 1-forms.**
Let $F : M \to N$ be a smooth map. Then we can define,

$$F^* : C^\infty(M) \to C^\infty(M)$$
$$F^*(g) = g \circ F.$$

And note that the pushforward of tangent vectors is essentially

$$F_* : T_pM \to T_{F(p)}N$$
$$F_*(v)(f_p) = v(f \circ F)_p) = v\left((F^*f)_p\right).$$

Then we have the dual map running in the other direction,

$$F^* : T^*_{F(p)}N \to T^*_p(M)$$
$$F^*w(v) = w(F_*(v)).$$

as the pullback of covectors.
So, if $w$ is a 1-form on $N$, define $F^*(w)$ to be the 1-form on $M$ given by,

$$\left(F^*w\right)_p = F^*(w_{F(p)}) \in T_pM.$$

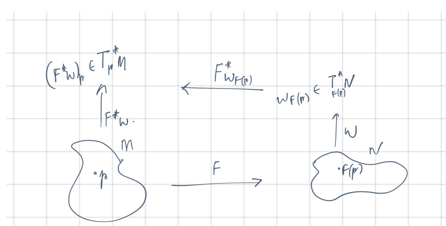



FIGURE 64

Basically there is an asymmetry in the defintion of *functions.* In case of vector fields, pushforwards are well defined under special conditions like diffeomorphisms (or F-relatedness). But here the pull back of 1-forms is natural.
Recall, if $g \in C^\infty(N)$, $dg \in \Omega^1(N)$ and $dg = \frac{\partial g}{\partial x_i}dx_i$ in the local coordintes $(U, x_1, \ldots, x_n)$ of $N$. What's the pull back of this 1-form?
Let $v \in T_pM$.

$$\begin{aligned}(F^*(dg))_p(v) &= dg|_{F(p)}(F_*(v)) \\ &= F_*(v)\left(g_{F(p)}\right) = v(g \circ F) \\ &= d(g\circ F)|_p(v) \equiv d(F^*g)|_p(v).\end{aligned}$$

So,

$$F^*(dg) = d(F^*g) = d(g \circ F) \in \Omega^1(M).$$

Pull back and differential commute with each other.

**Lemma 5.6.** *If $w_1, w_2$ are 1-forms on $N$ and $g \in C^\infty(N)$ then,*

(1) $F^*(w_1 + w_2) = F^*w_1 + F^*w_2$.
(2) $F^*(gw_1) = F^*g(F^*w_1) = (g \circ F)(F^*w_1)$

These follow from the linearity of $F$ at the level of co-tangent spaces.

**Proposition 5.7.** *If $w$ is a smooth 1-form on $N$, then $F^*w$ is a smooth 1-form on $M$.*

*Proof.* Let $p \in M$, take a chart $(V, y_1, \ldots, y_n)$ of $N$ containing $F(p)$ and a chart $(U, x_1, \ldots, x_n)$ of $M$ containing $p$ such that $F(U) \subseteq V$.

Then locally, $w|_V = a_i dy_i, a_i \in C^\infty(N)$. and

$$\begin{aligned}F^*w|_U &= F^*(a_i dy_i) \\ &= (a_i \circ F)\, d(y_i \circ F) \\ &= (a_i \circ F)\left(\frac{\partial F_i}{\partial x_j} dx_j\right).\end{aligned}$$

which is smooth. Thus $F^*w|_U$ is smooth. Since, smoothness is a lcoal property, $F^*w$ is smooth on $M$. $\square$

So, we have the pullback of smooth 1-forms,

$$F^*\Omega^1(N) \to \Omega^1(M)$$

5.1.1. *Restriction of 1-forms to regular submanifolds.* Let V be a vector space and $W \subseteq V$ as linear subspace. If $\alpha \in V^*$, $\alpha|_W \in W^*$.

If $S \subseteq M$ is a regular submanifold, then for any $p \in S, T_pS \subseteq T_pM$.

This way, if $v \in T^*M$ then, $v|_{T_pS} \in T^*S$.

This is a surjection not an injection.

If $w$ is a 1-form on $M$, $w|_S$ is the 1-form on S defined for any $p \in S$ as,

$$(w|_s)_p = w_p|_{T_pS}.$$

Note that the inclusion map $i : S \hookrightarrow M$ is an embedding. Then, $i^*w = w|_s$

$$i^*w = w|_S$$

where $(i^*w)_p(v) = w_p(i_*v) = w_p(v)$.

We will often make an abuse of notation and write the induced form also as $w$. So, if $w \in \Omega^1(M)$, then $i^*w \in \Omega^1(S)$.

Noting for $M = \mathbb{R}^n$, that any $w \in \Omega^1(\mathbb{R}^n)$ can be written as $w = a_1dx_1 + \cdots + a_ndx_n$ where $a_i \in C^\infty(M)$ let's see some examples.

**Examples.**

1. Consider $S^2 \subseteq \mathbb{R}^3$, and the 1-form $dz \in \Omega^1(\mathbb{R}^3)$. Then $w = dz|_{S^2} \in \Omega^1(S^2)$. This induced 1-form vanishes at the North and South poles!

Consider the steregraphics charts on $S^2$. And let $p = (0,0,1)$, the north pole.

Then $T_pS^2$ is a subspace of $T_p\mathbb{R}^3$ spanned by $\left\{\frac{\partial}{\partial x}|_p\right\}, \frac{\partial}{\partial y}|_p$.

So, $w_p\left(\frac{\partial}{\partial x}|_p\right) = dz|_p\left(\frac{\partial}{\partial x}|_p\right) = 0$ and so is $w_p\left(\frac{\partial}{\partial y}|_p\right)$. That is, $w_p = 0$.

Similarly $w$ vanishes at $(0,0,-1)$. This is precisely connected to the vanishing vector fields on $S^2$.

2. Consider $S^1 \subseteq \mathbb{R}^2$. It is easy to see that, $w_1 = dx|_{S^1}$ vanishes at $(\pm 1, 0)$ and $w_2 = dy|_{S^1}$ vanishes at $(0, \pm 1)$.

These are basically examples of $df$ (or $f_*$) vanishing at all the critical points of $f \in C^\infty(M)$. Any 1-form can be written trivially as the differential of a function.

Note that there exists a global frame on $S^1$ that is non-vanishing everywhere! Consider, $w = -ydx + xdy \in \Omega^1(\mathbb{R}^2)$ and induce it on $S^1$. We know that $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$ is non-vanishing. So, $w|_p(X_p) = x^2 + y^2 = 1 \forall p \in S^1$ and forms a global frame of $T^*S^1$. This means $T^*S^1$ is the trivial bundle $S^1 \times \mathbb{R}$ (as it must be since $TS^1$ is)!

3. Consider $q : \mathbb{R} \to S^1, q(t) = (\cos t, \sin t)$. Let's find the pullback of $w = -ydx + xdy$ onto $\mathbb{R}$.

$$\begin{aligned}
q^*(w) &= q^*(-ydx) + q^*(xdy) \\
&= -(y \circ q)\, d(x \circ q) + (x \circ q)\, d(y \circ q) \\
&= -\sin t d(\cos t) + \cos t\,(d(\sin t)) \\
&= \sin^2 tdt + \cos^2 tdt = dt.
\end{aligned}$$

This is dual to $\frac{\partial}{\partial t}$ whose pushforward makes sense because it's a $\mathbb{Z}$ invariant vector field on $\mathbb{R}$ and is pushed precisely to the non-vanishing vector field $w$.

To develop differential forms further, we need some multi linear algebra.

5.1.2. *Some (multi) linear algebra ...* Let $V$ be a vector space.

$$\alpha : V \times \cdots \times V \to \mathbb{R}$$

is called k-multilinear (over $\mathbb{R}$ for our purposes) if $\alpha$ is linear in each variable.

We denote $\mathcal{T}^k(V)$ as the set of all k-linear maps on $V$. Note that, $\mathcal{T}^1(V) = V^*$ and $\mathcal{T}^k(V) \cong V^* \otimes \cdots \otimes V^*$ which defines the multilinear function,

$$\phi_1 \otimes \cdots \otimes \phi_k \, (v_1, \ldots, v_k) = \phi_1(v_1) \cdots \phi_k(v_k)$$

.

**Proposition 5.8.** *If* $v_1, \ldots, v_n$ *is a basis of* $V$ *and* $v_1^*, \ldots, v_n^*$ *is the dual basis of* $V^*$*, then* $v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*$*,* $i_1, \ldots, i_k \in \{1, \ldots, n\}$ *is the basis of* $\mathcal{T}^k(V)$*. And thus, dim* $\mathcal{T}^k(V) = n^k$.

## Appendix A. Tips of the Icebergs

A.1. **Connected components of manifolds.** Examples of manifolds with each conected component having different dimension!

A.2. **Homology Groups.**

A.3. **Open Mapping Theorem.** ft. Vamsi Pritham Pingali

A.4. **Smooth Structures on manifolds.**

## Appendix B. Basic Point Set Topology

To be TEXed.

## Appendix C. Solutions to Problem Sets